Nuclear Physics as Precision Science

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- A brief introduction to nuclear interactions
- Basics of nuclear lattice simulations
- Results from nuclear lattice simulations
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- New insights into nuclear clustering
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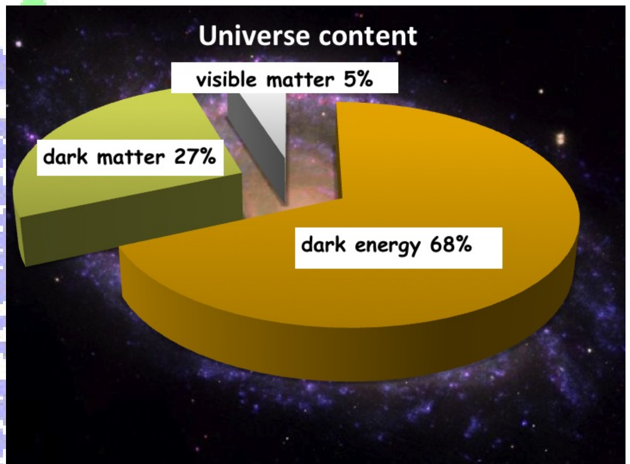
- Summary & outlook

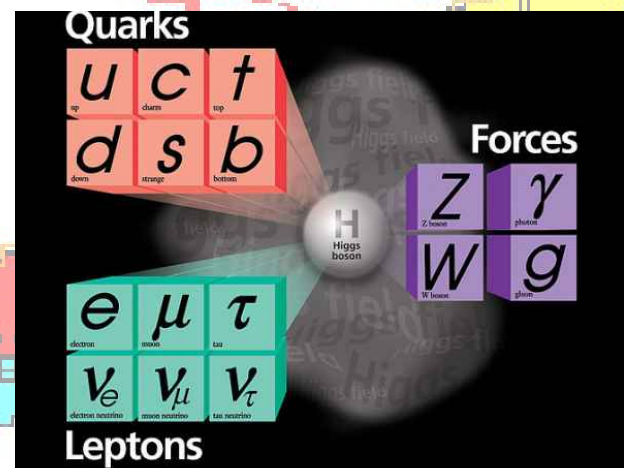
The BIG Picture

WHY NUCLEAR PHYSICS?

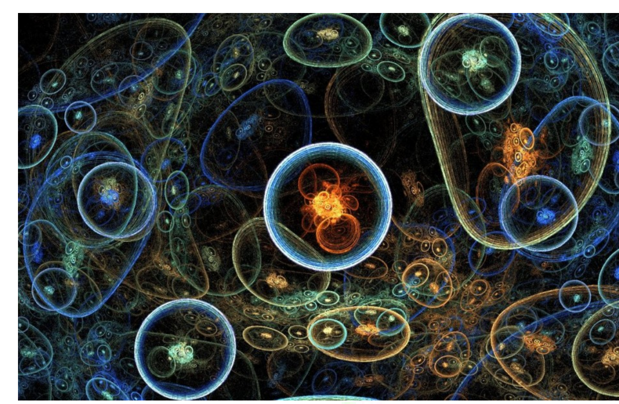
- **The matter we are made off**



- **The last frontier of the SM**



- **Access to the Multiverse**



Proton Number Z

Neutron Number N

AB INITIO NUCLEAR STRUCTURE and SCATTERING

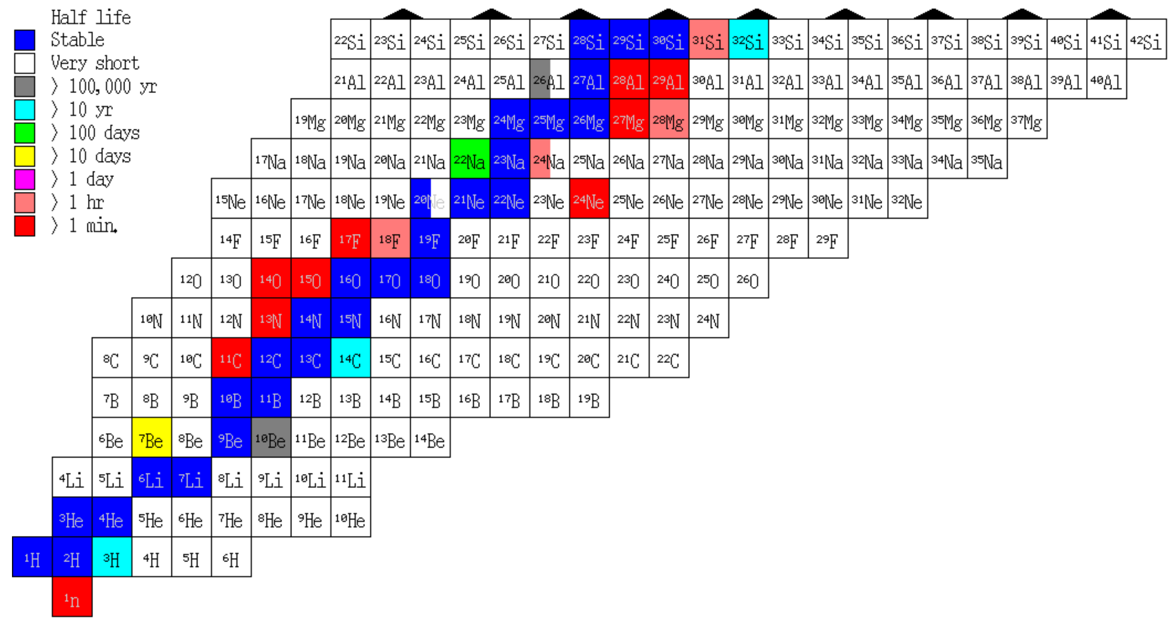
- Nuclear structure:

  ⋆ 3-nucleon forces

  ⋆ limits of stability

  ⋆ alpha-clustering

  ⋮



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- Nuclear scattering: processes relevant for nuclear astrophysics

  ⋆ alpha-particle scattering: $^4\text{He} + {}^4\text{He} \to {}^4\text{He} + {}^4\text{He}$

  ⋆ triple-alpha reaction: $^4\text{He} + {}^4\text{He} + {}^4\text{He} \to {}^{12}\text{C} + \gamma$

  ⋆ alpha-capture on carbon: $^4\text{He} + {}^{12}\text{C} \to {}^{16}\text{O} + \gamma$

  ⋮

THE NUCLEAR LANDSCAPE: AIMS & METHODS

- Theoretical methods:

– Lattice QCD: *A = 0,1,2, . . .*

– NCSM, Faddeev-Yakubowsky, GFMC, ... : *A = 3 - 16*

– coupled cluster, . . .: *A = 16 -100*

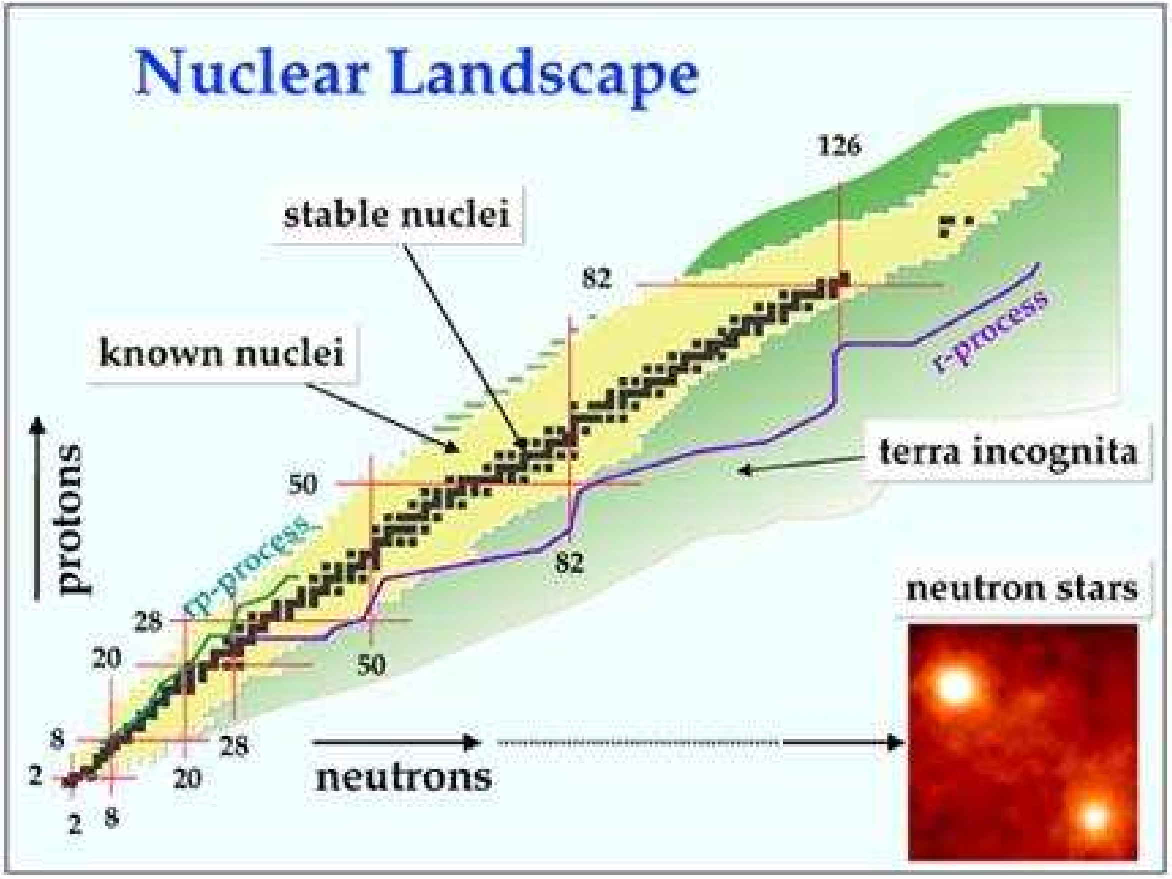
– density functional theory, . . .: $A \geq 100$

- Chiral EFT:

– provides **accurate 2N, 3N and 4N forces**

– successfully applied in light nuclei with *A = 2, 3, 4*

– combine with simulations to get to larger A

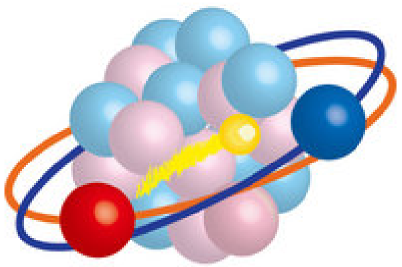


⇒ Chiral Nuclear Lattice Effective Field Theory

A brief introduction to nuclear interactions

NUCLEAR PHSICS: A PRIMER

- Nuclei are self-bound system of fermions (protons & neutrons)
- Bound by the **strong** force
- Repulsion also from the **Coulomb** force

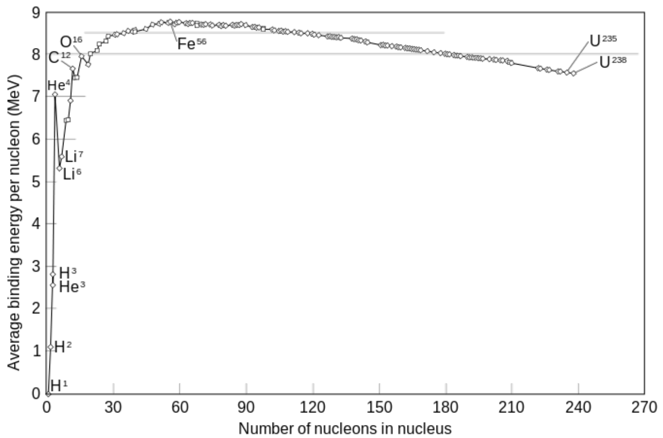


- Non-relativistic system:

$$H_{\text{nuclear}} = T + V$$

- Nuclear Hamiltonian:

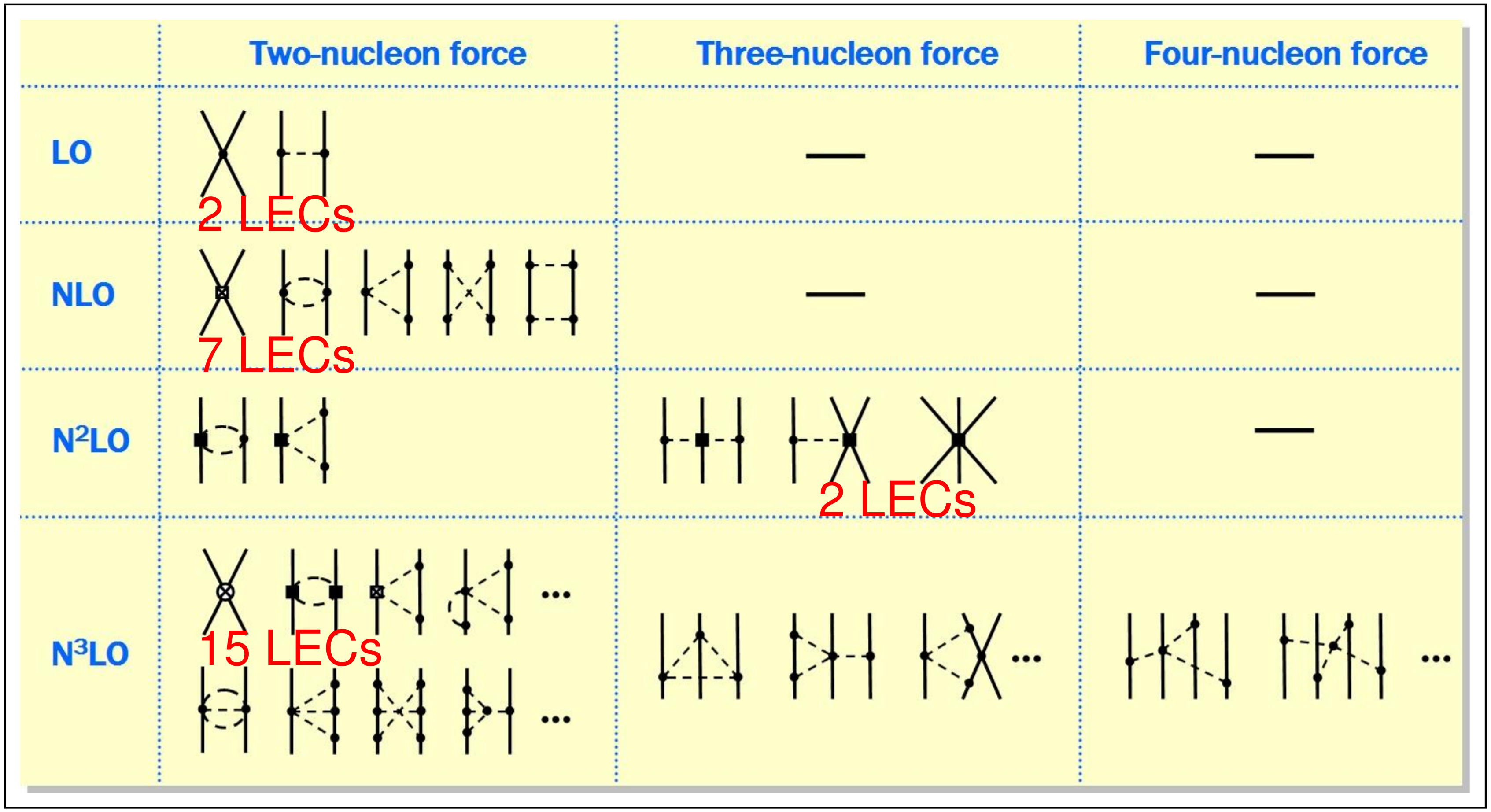
$$V = V_{NN} + V_{NNN} + \ldots$$



- Dominant two-nucleon potential V_{NN},
  but small three-nucleon force V_{NNN} is required
- Nuclear binding energies $\ll$ nuclear masses
- The nuclear Hamiltonian can be **systematically** analyzed
  using the **symmetries** of the strong interactions

Weinberg 1990, 1991

CHIRAL POTENTIAL and NUCLEAR FORCES



$\mathcal{O}((Q/\Lambda_\chi)^0)$

$\mathcal{O}((Q/\Lambda_\chi)^2)$

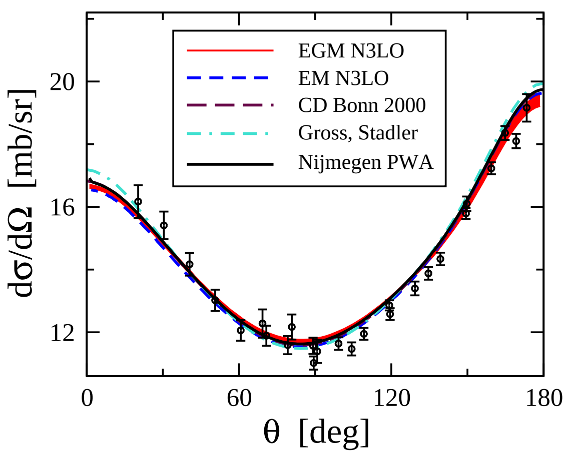
$\mathcal{O}((Q/\Lambda_\chi)^3)$

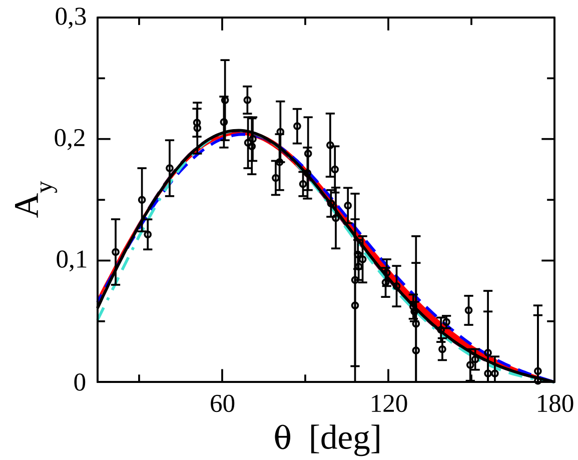
$\mathcal{O}((Q/\Lambda_\chi)^4)$

- explains naturally the observed hierarchy of nuclear forces
- MANY successful tests in few-nucleon systems (continuum calc's)

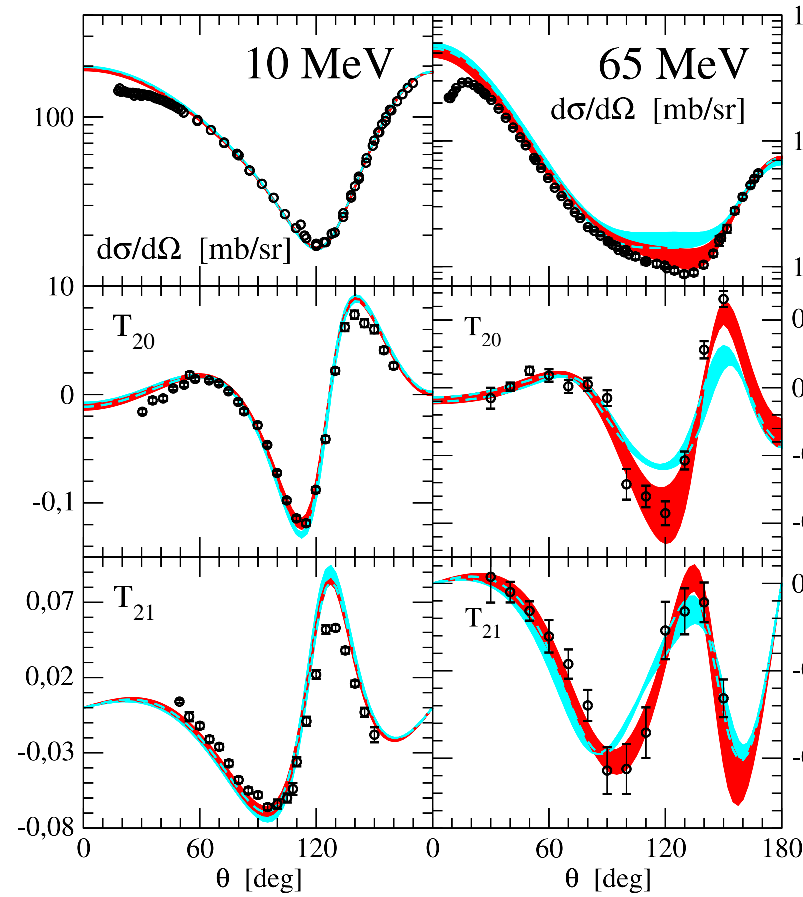
RESULTS at N3LO

- np scattering

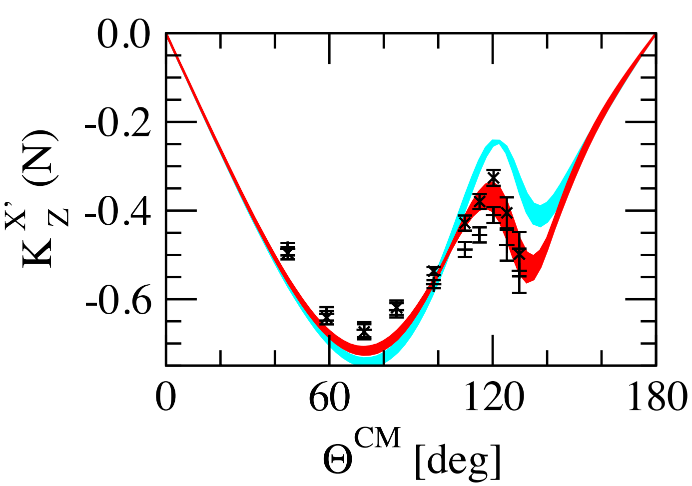




- nd scattering



- pol. transfer in pd scattering



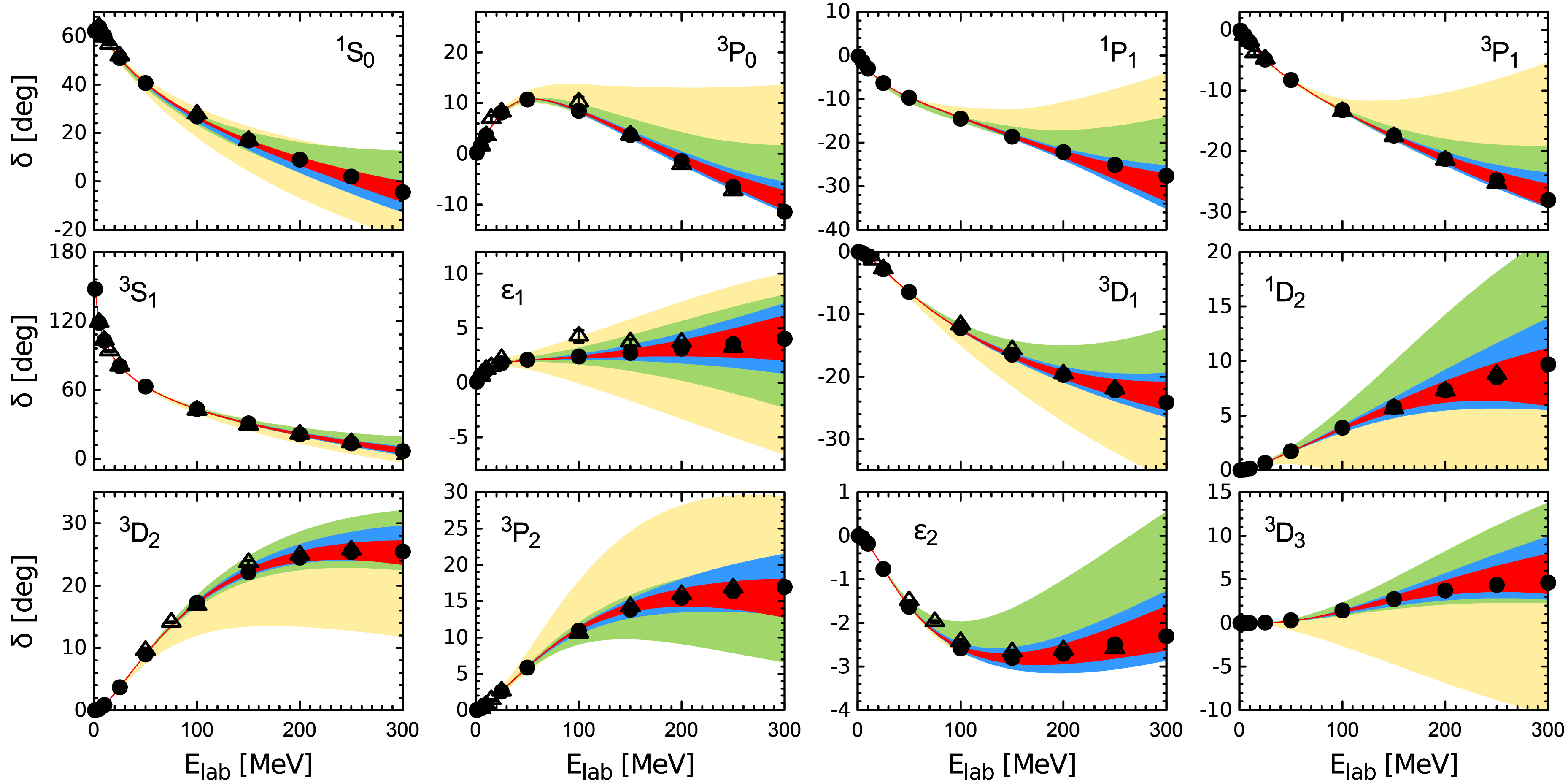
Epelbaum, Hammer, UGM,
Rev. Mod. Phys. **81** (2009) 1773

PHASE SHIFTS at N4LO

- N4LO analysis, better error estimates

Epelbaum, Krebs, UGM, Phys. Rev. Lett. **115** (2015) 122301

- Precision phase shifts with small uncertainties up to $E_{\rm lab} = 300$ MeV



NLO N2LO N3LO N4LO

Basics of nuclear lattice simulations

for an easy intro, see: UGM, Nucl. Phys. News **24** (2014) 11

NUCLEAR LATTICE EFFECTIVE FIELD THEORY

Frank, Brockmann (1992), Koonin, Müller, Seki, van Kolck (2000) , Lee, Schäfer (2004), . . .
Borasoy, Krebs, Lee, UGM, Nucl. Phys. **A768** (2006) 179; Borasoy, Epelbaum, Krebs, Lee, UGM, Eur. Phys. J. **A31** (2007) 105

- *new method* to tackle the nuclear many-body problem

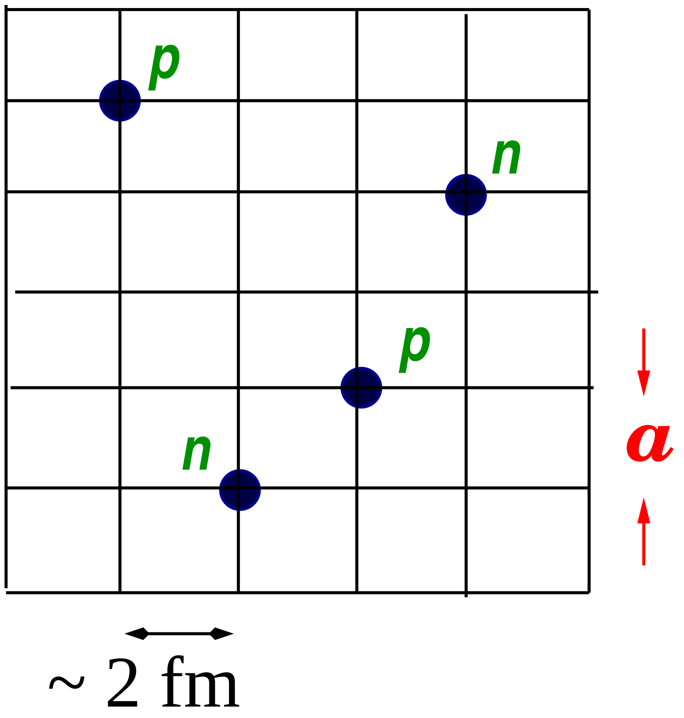
- discretize space-time $V = L_s \times L_s \times L_s \times L_t$:
  nucleons are point-like particles on the sites

- discretized chiral potential w/ pion exchanges
  and contact interactions + Coulomb

$\rightarrow$ see Epelbaum, Hammer, UGM, Rev. Mod. Phys. **81** (2009) 1773

- typical lattice parameters

$$p_{\max} = \frac{\pi}{a} \simeq 314 \text{ MeV [UV cutoff]}$$



- strong suppression of sign oscillations due to approximate Wigner SU(4) symmetry

E. Wigner, Phys. Rev. **51** (1937) 106; T. Mehen et al., Phys. Rev. Lett. **83** (1999) 931; J. W. Chen et al., Phys. Rev. Lett. **93** (2004) 242302

- physics independent of the lattice spacing for $a = 1 \ldots 2$ fm

J. Alarcon et al., EPJA **53** (2017) 83

TRANSFER MATRIX METHOD

- Correlation–function for A nucleons: $Z_A(\tau) = \langle \Psi_A | \exp(-\tau H) | \Psi_A \rangle$

  Euclidean time

  with Ψ_A a Slater determinant for A free nucleons
  [or a more sophisticated (correlated) initial/final state]

- Transient energy

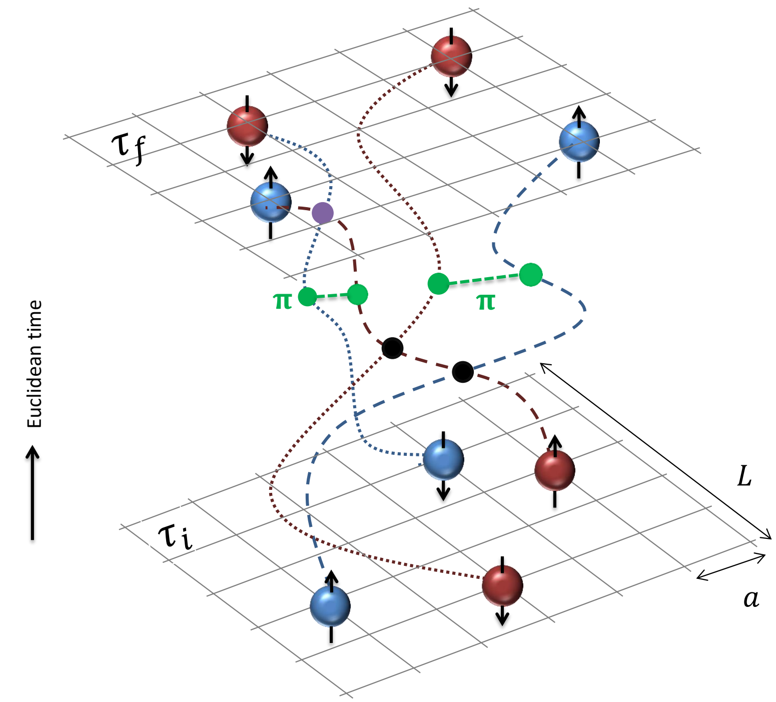
$$E_A(\tau) = -\frac{d}{d\tau} \ln Z_A(\tau)$$

→ ground state: $E_A^0 = \lim_{\tau\to\infty} E_A(\tau)$

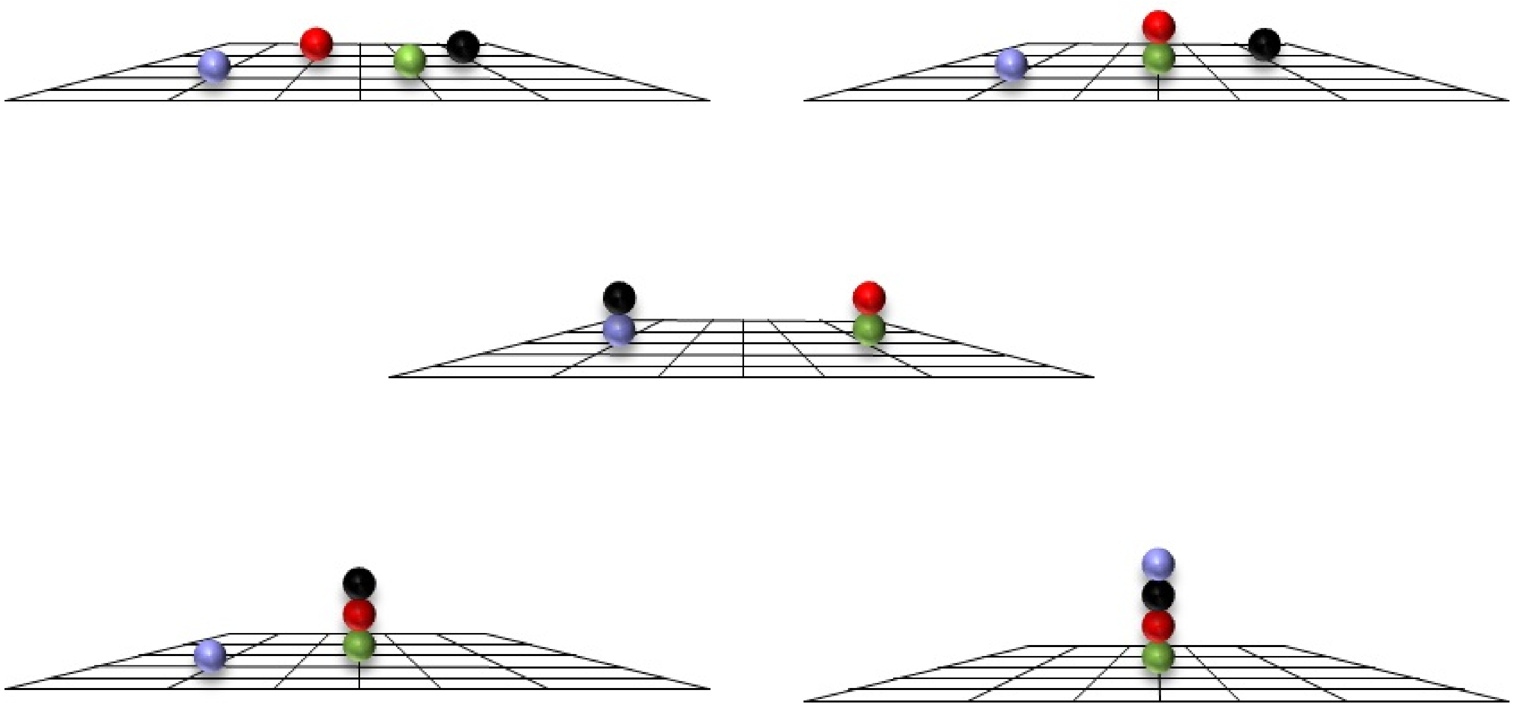
- Exp. value of any normal–ordered operator $\mathcal{O}$

$$Z_A^{\mathcal{O}} = \langle \Psi_A | \exp(-\tau H/2)\, \mathcal{O}\, \exp(-\tau H/2)\, | \Psi_A \rangle$$

$$\lim_{\tau\to\infty} \frac{Z_A^{\mathcal{O}}(\tau)}{Z_A(\tau)} = \langle \Psi_A | \mathcal{O} \, | \Psi_A \rangle$$



CONFIGURATIONS



⇒ all *possible* configurations are sampled

⇒ preparation of *all possible* initial/final states

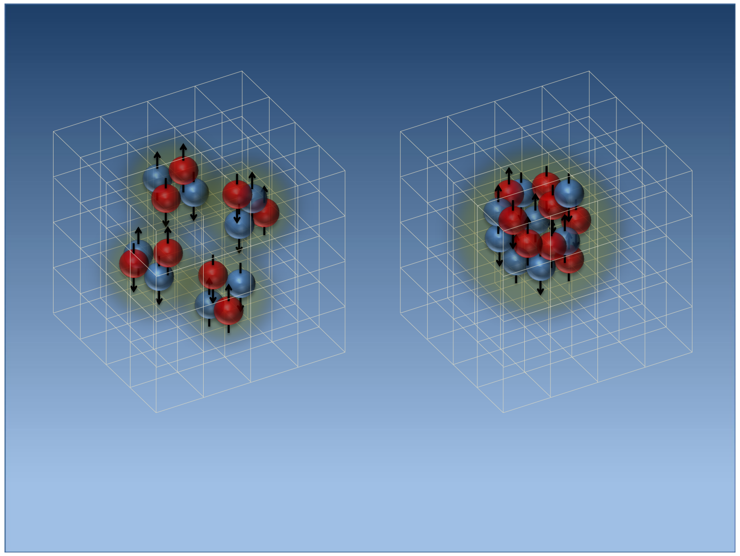
⇒ *clustering* emerges *naturally*

COMPUTATIONAL EQUIPMENT

- Present = JUQUEEN (BlueGene/Q)

6 Pflops

Lattice: some results



NLEFT

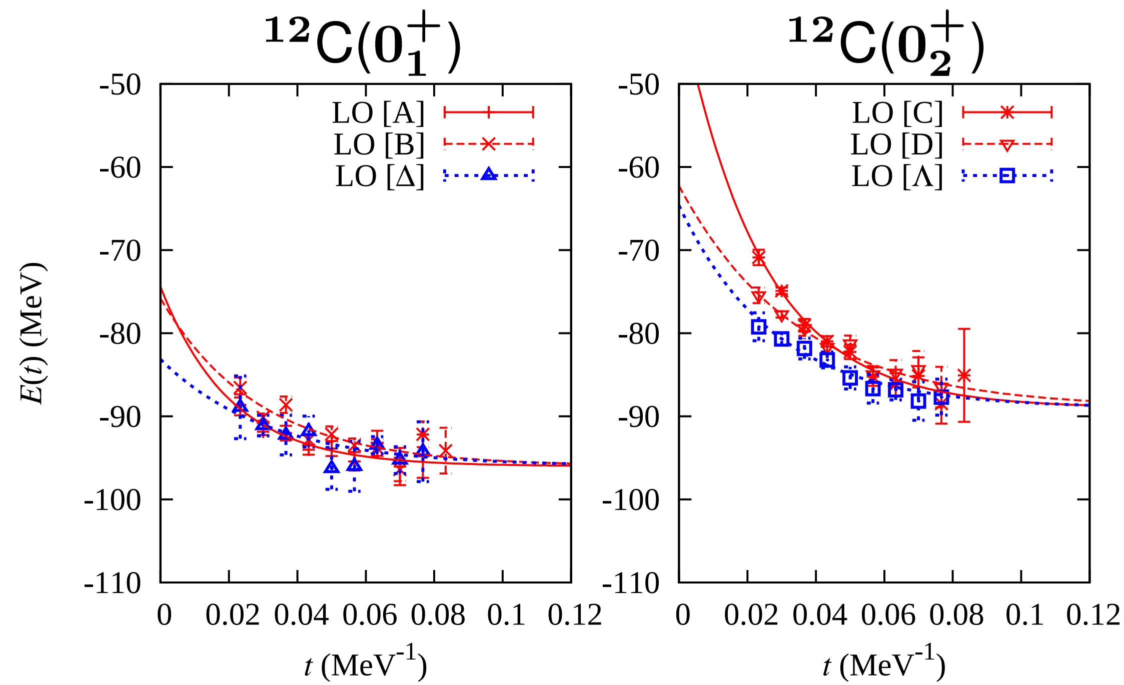
Epelbaum, Krebs, Lähde, Lee, Luu, UGM, Rupak + post-docs + students

FIXING PARAMETERS and FIRST RESULTS

Epelbaum, Krebs, Lee, UGM, Phys. Rev. Lett. **104** (2010) 142501; Eur. Phys. J. A **45** (2010) 335; . . .

- some groundstate energies and differences [NNLO, 11+2 LECs]

	E [MeV]	NLEFT	Exp.
old algorithm	^{3}He - ^{3}H	0.78(5)	0.76
	^{4}He	−28.3(6)	−28.3
	^{8}Be	−55(2)	−56.5
	^{12}C	−92(3)	−92.2
new algorithm	^{16}O	−131(1)	−127.6
	^{20}Ne	−166(1)	−160.6
	^{24}Mg	−198(2)	−198.3
	^{28}Si	−234(3)	−236.5



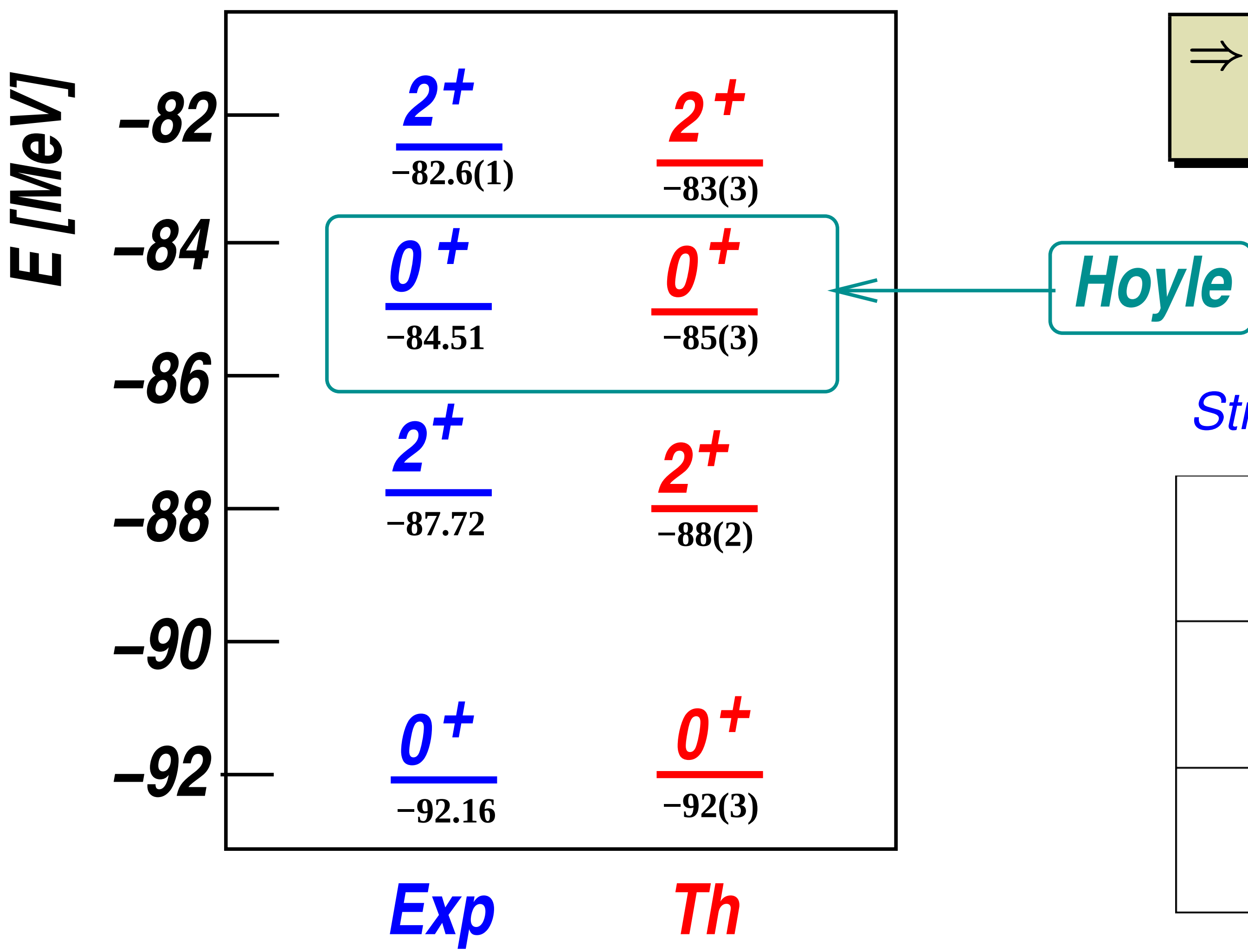
- promising results ⇒ uncertainties down to the 1% level
- excited states more difficult ⇒ projection MC method + triangulation

BREAKTHROUGH: SPECTRUM of CARBON-12

Epelbaum, Krebs, Lee, UGM, Phys. Rev. Lett. 106 (2011) 192501

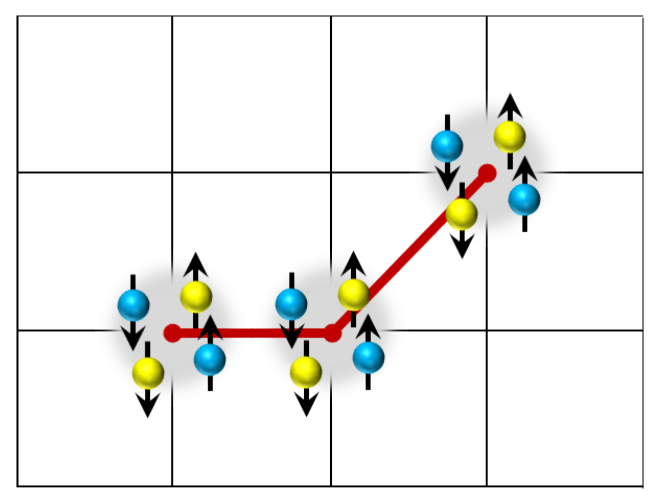
Epelbaum, Krebs, Lähde, Lee, UGM, Phys. Rev. Lett. 109 (2012) 252501

- After $8 \cdot 10^6$ hrs JUGENE/JUQUEEN (and "some" human work)



$\Rightarrow$ First ab initio calculation of the Hoyle state $\surd$

Structure of the Hoyle state:



A SHORT HISTORY of the HOYLE STATE

- Heavy element generation in massive stars: triple-α process

Bethe 1938, Öpik 1952, Salpeter 1952, Hoyle 1954, ...

$$^4\text{He} + {}^4\text{He} \rightleftharpoons {}^8\text{Be}$$
$$^8\text{Be} + {}^4\text{He} \rightleftharpoons {}^{12}\text{C}^* \rightarrow {}^{12}\text{C} + \gamma$$
$$^{12}\text{C} + {}^4\text{He} \rightleftharpoons {}^{16}\text{O} + \gamma$$

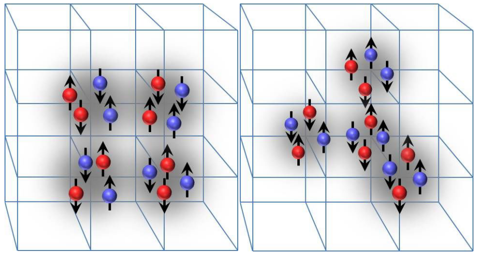
- Hoyle's contribution: calculation of the relative abundances of ^{4}He, ^{12}C and ^{16}O
  $\Rightarrow$ need a resonance close to the ^{8}Be + ^{4}He threshold at $E_R \simeq 0.37$ MeV
  $\Rightarrow$ this corresponds to a $J^P = 0^+$ excited state 7.7 MeV above the g.s.

- a corresponding state was experimentally confirmed at Caltech at
  $E - E(\text{g.s.}) = 7.653 \pm 0.008$ MeV   Dunbar et al. 1953, Cook et al. 1957

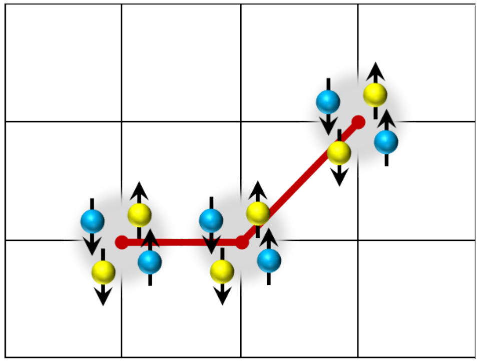
- still on-going experimental activity, e.g. EM transitions at SDALINAC
  M. Chernykh et al., Phys. Rev. Lett. 98 (2007) 032501

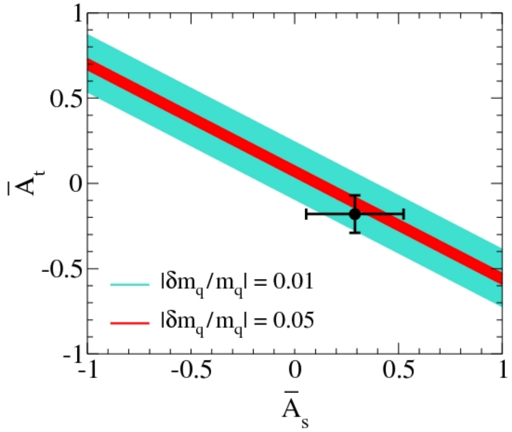
- side remark: relevance to the antrhopic principle?
  H. Kragh, An anthropic myth: Fred Hoyle's carbon-12 resonance level,
  Arch. Hist. Exact Sci. 64 (2010) 721

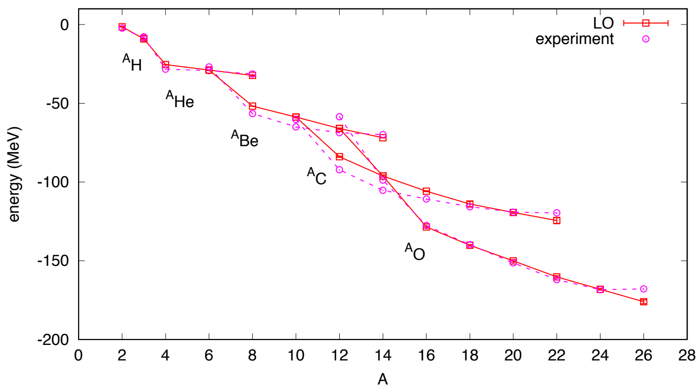
RESULTS from LATTICE NUCLEAR EFT

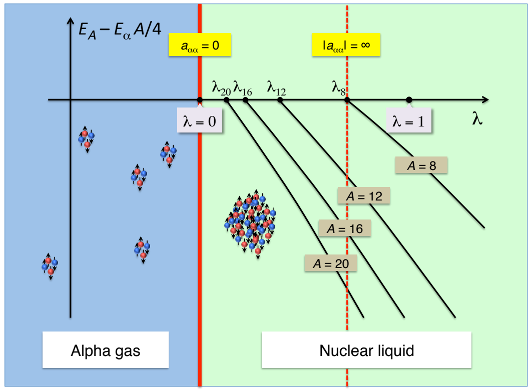


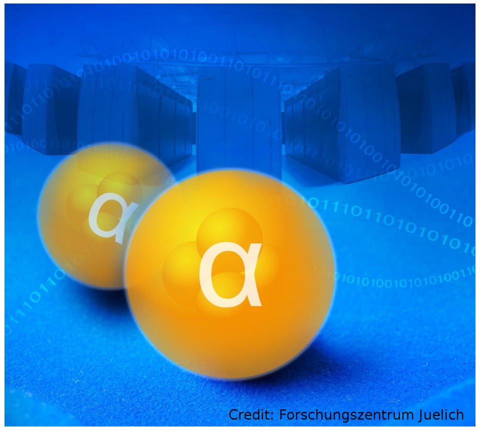
- ☐ Lattice EFT calculations for A=3,4,6,12 nuclei, PRL 104 (2010) 142501
- ☐ *Ab initio* calculation of the Hoyle state, PRL 106 (2011) 192501
- ☐ Structure and rotations of the Hoyle state, PRL 109 (2012) 142501
- ☐ Validity of Carbon-Based Life as a Function of the Light Quark Mass PRL 110 (2013) 142501
- ☐ *Ab initio* calculation of the Spectrum and Structure of ^{16}O, PRL 112 (2014) 142501
- ☐ *Ab initio* alpha-alpha scattering, Nature 528 (2015) 111
- ☐ Nuclear Binding Near a Quantum Phase Transition, PRL 117 (2016) 132501
- ☐ *Ab initio* calculations of the isotopic dependence of nuclear clustering, arXiv:1702.05177



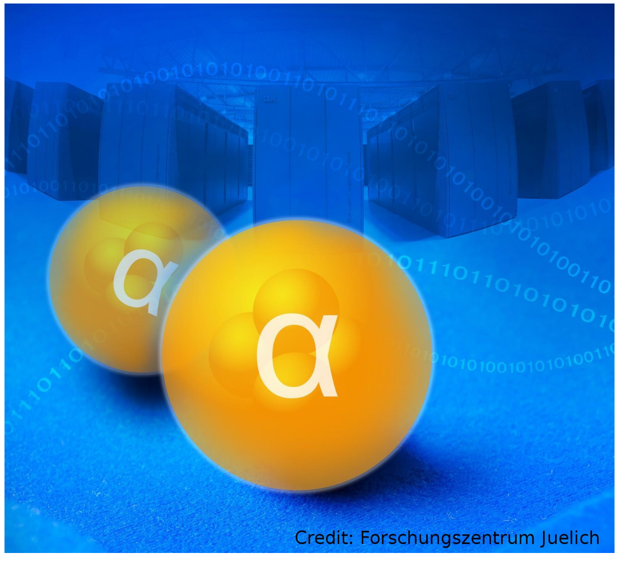








Ab initio calculation of α-α scattering

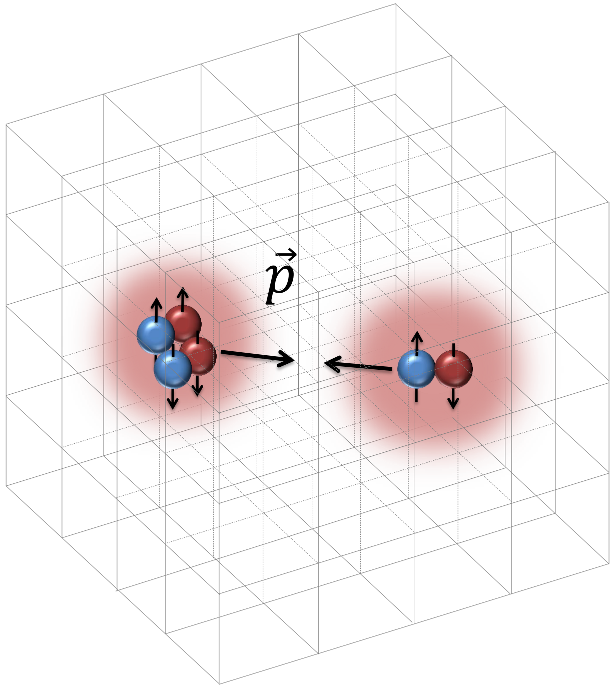


Elhatisari, Lee, Rupak, Epelbaum, Krebs, Lähde, Luu, UGM,
Nature **528** (2015) 111 [arXiv:1506.03513]

NUCLEUS–NUCLEUS SCATTERING on the LATTICE

- Processes involving α-particles and α-type nuclei comprise a major part of stellar nucleosynthesis, and control the production of certain elements in stars

- Ab initio calculations of scattering and reactions suffer from computational scaling with the number of nucleons in the clusters



Lattice EFT computational scaling $\Rightarrow (A_1 + A_2)^2$

Rupak, Lee, Phys. Rev. Lett. **111** (2013) 032502
Pine, Lee, Rupak, Eur. Phys. J. A **49** (2013) 151
Elhatisari, Lee, Phys. Rev. C **90** (2014) 064001
Elhatisari et al., Phys.Rev. C **92** (2015) 054612
Elhatisari, Lee, UGM, Rupak, Eur. Phys. J. A **52** (2016) 174

ADIABATIC PROJECTION METHOD

- Basic idea to treat scattering and inelastic reactions:
  split the problem into two parts

First part:
use Euclidean time projection to construct an *ab initio* low-energy cluster Hamiltonian, called the **adiabatic Hamiltonian**

Second part:
compute the two-cluster scattering phase shifts or reaction amplitudes using the adiabatic Hamiltonian

ADIABATIC PROJECTION METHOD II

- Construct a low-energy effective theory for clusters

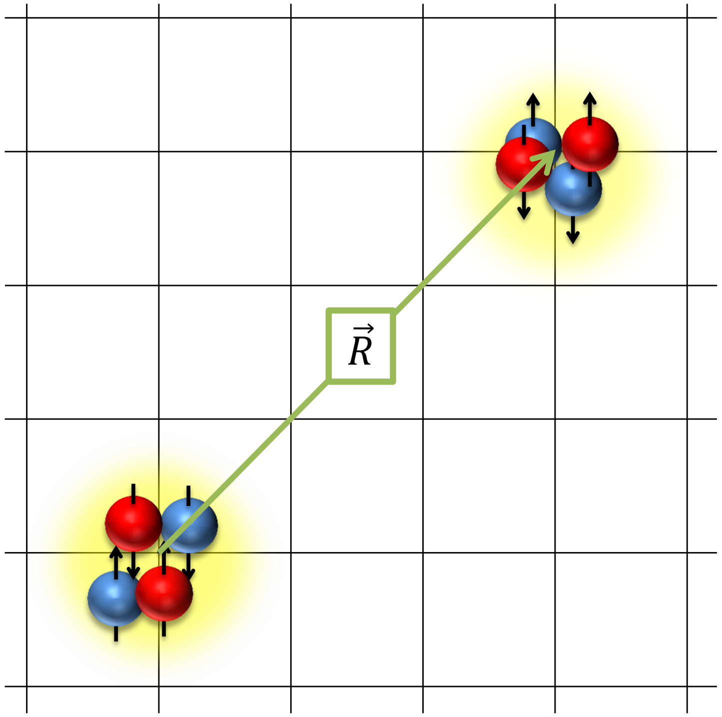
- Use initial states parameterized by the relative separation between clusters

$$|\vec{R}\rangle = \sum_{\vec{r}} |\vec{r} + \vec{R}\rangle \otimes \vec{r}$$

- project them in Euclidean time with the chiral EFT Hamiltonian H

$$|\vec{R}\rangle_\tau = \exp(-H\tau)|\vec{R}\rangle$$

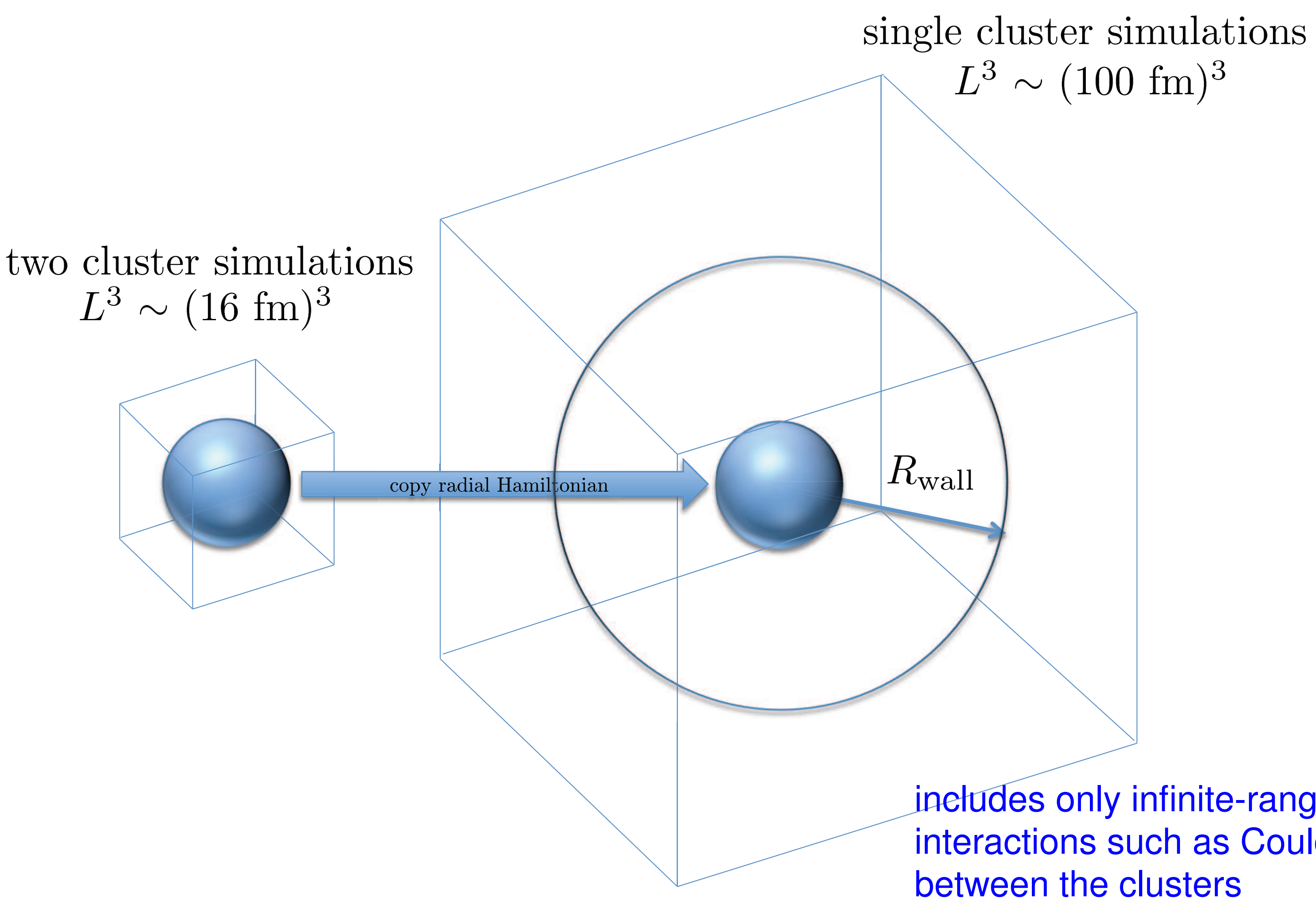
$\rightarrow$ "dressed cluster states" (polarization, deformation, Pauli)



- Adiabatic Hamiltonian (requires norm matrices)

$$[H_\tau]_{\vec{R}\vec{R}'} = {}_\tau\langle\vec{R}|H|\vec{R}'\rangle_\tau$$

ADIABATIC HAMILTONIAN plus COULOMB



includes only infinite-range interactions such as Coulomb between the clusters

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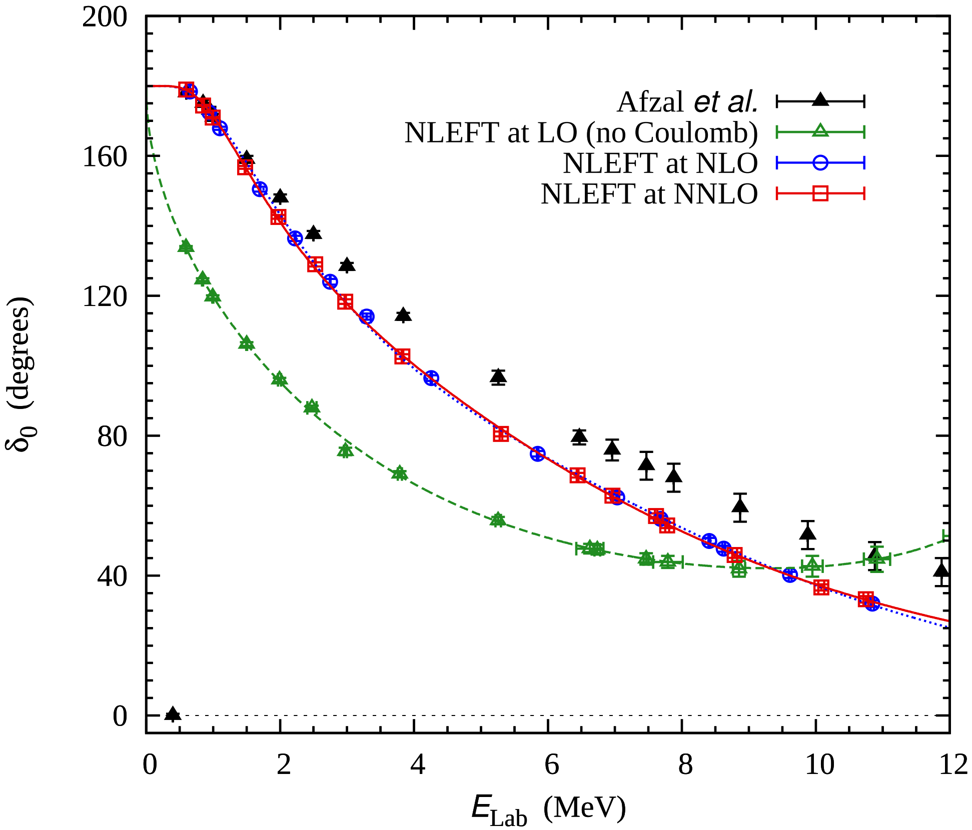
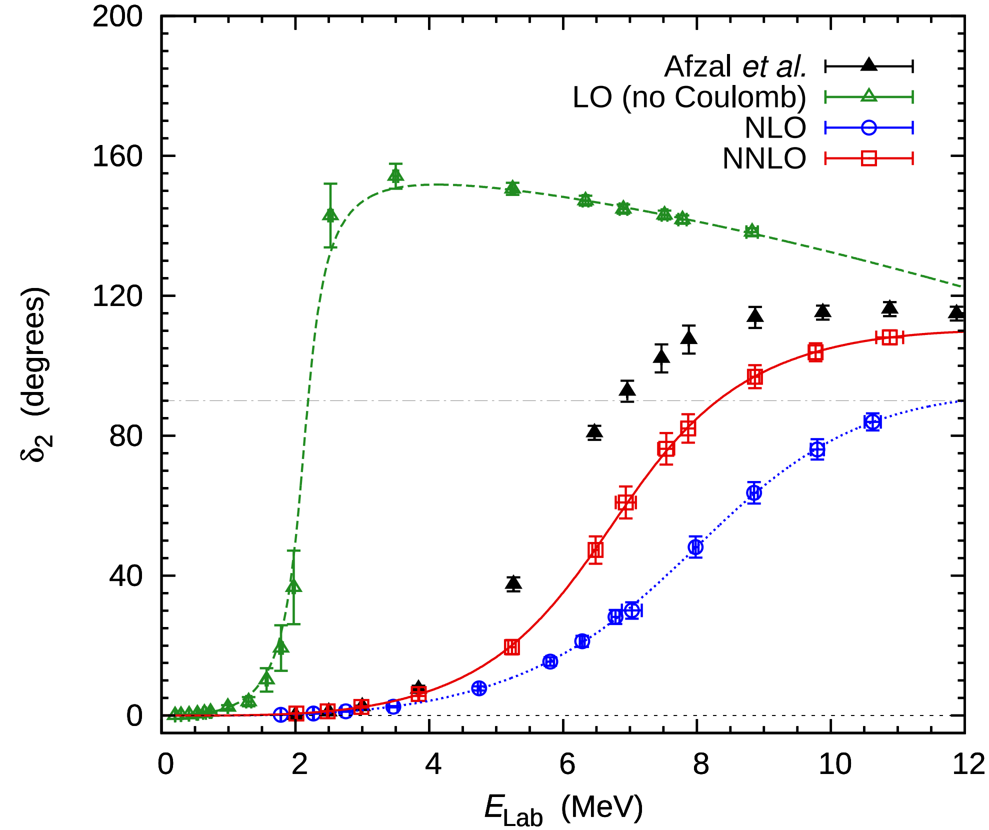
PHASE SHIFTS

- Same NNLO Lagrangian as used for the study of ^{12}C and ^{16}O





$$E_R^{\rm NNLO} = -0.11(1)\ \text{MeV}\quad [+0.09\ \text{MeV}]$$

$$E_R^{\rm NNLO} = 3.27(12)\ \text{MeV}\quad [\,2.92(18)\ \text{MeV}]$$

$$\Gamma_R^{\rm NNLO} = 2.09(16)\ \text{MeV}\quad [1.35(50)\ \text{MeV}]$$

Data: Afzal et al., Rev. Mod. Phys. **41** (1969) 247

New insights into nuclear clustering

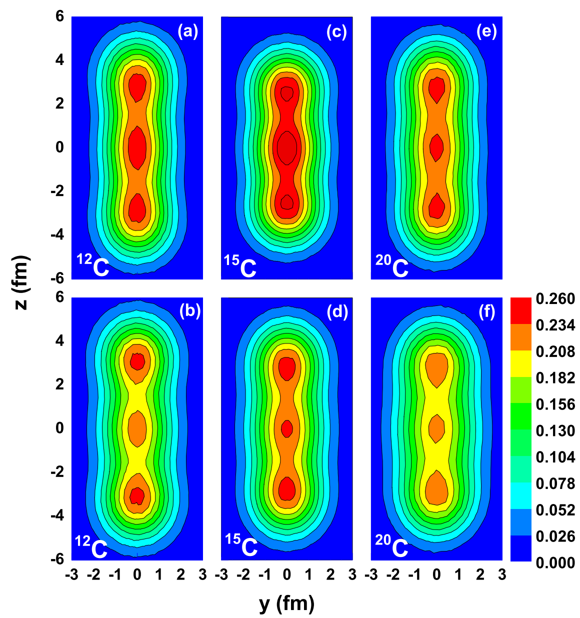
Elhatisari, Epelbaum, Krebs, Lähde, Lee, Li, Lu, UGM, Rupak
[arXiv:1702.05117]

CLUSTERING in NUCLEI

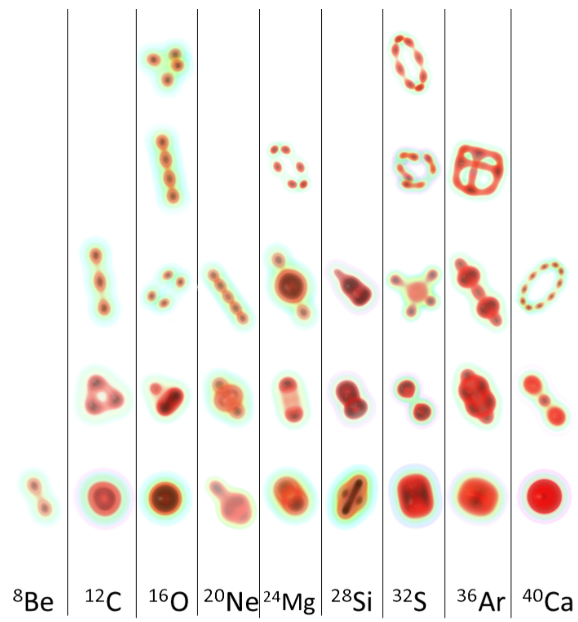
- Introduced theoretically by Wheeler already in 1937:

> John Archibald Wheeler, "Molecular Viewpoints in Nuclear Structure," Physical Review **52** (1937) 1083

- many works since then... Ikeda, Horiuchi, Freer, Schuck, Röpke, Khan, Zhou, Iachello, . . .



Zhao, Itagaki, Meng (2015)



Ebran, Khan, Niksic, Vretenar (2014)

α-clusters

$\Rightarrow$ can we understand this phenomenon from *ab initio* calculations?

EARLIER RESULTS on NUCLEAR CLUSTERING

- Already a number of intriguing results on clustering:

  Ab initio calculation of the spectrum and structure of ^{12}C (esp. the Hoyle state)

  Ab initio calculation of the spectrum and structure of ^{16}O

  Ground state energies of α-type nuclei up to ^{28}Si within 1%

  Ab initio calculation of α-α scattering

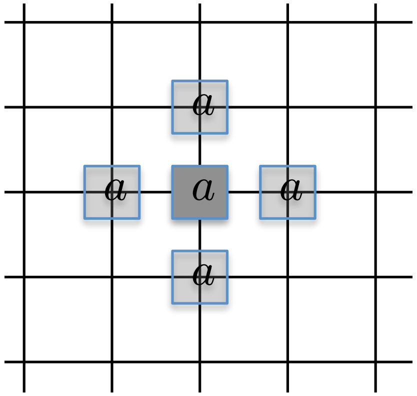
  Quantum phase transition from Bose gas of α's to nuclear liquid for α-type nuclei

- However: when adding extra neutrons/protons, the precision quickly deteriorates due to sign oscillations

- New LO action with smeared SU(4) local+non-local symmetric contact interactions & smeared one-pion exchange

$$a_{\rm NL}(\mathbf{n}) = a(\mathbf{n}) + s_{\rm NL} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a(\mathbf{n}')$$

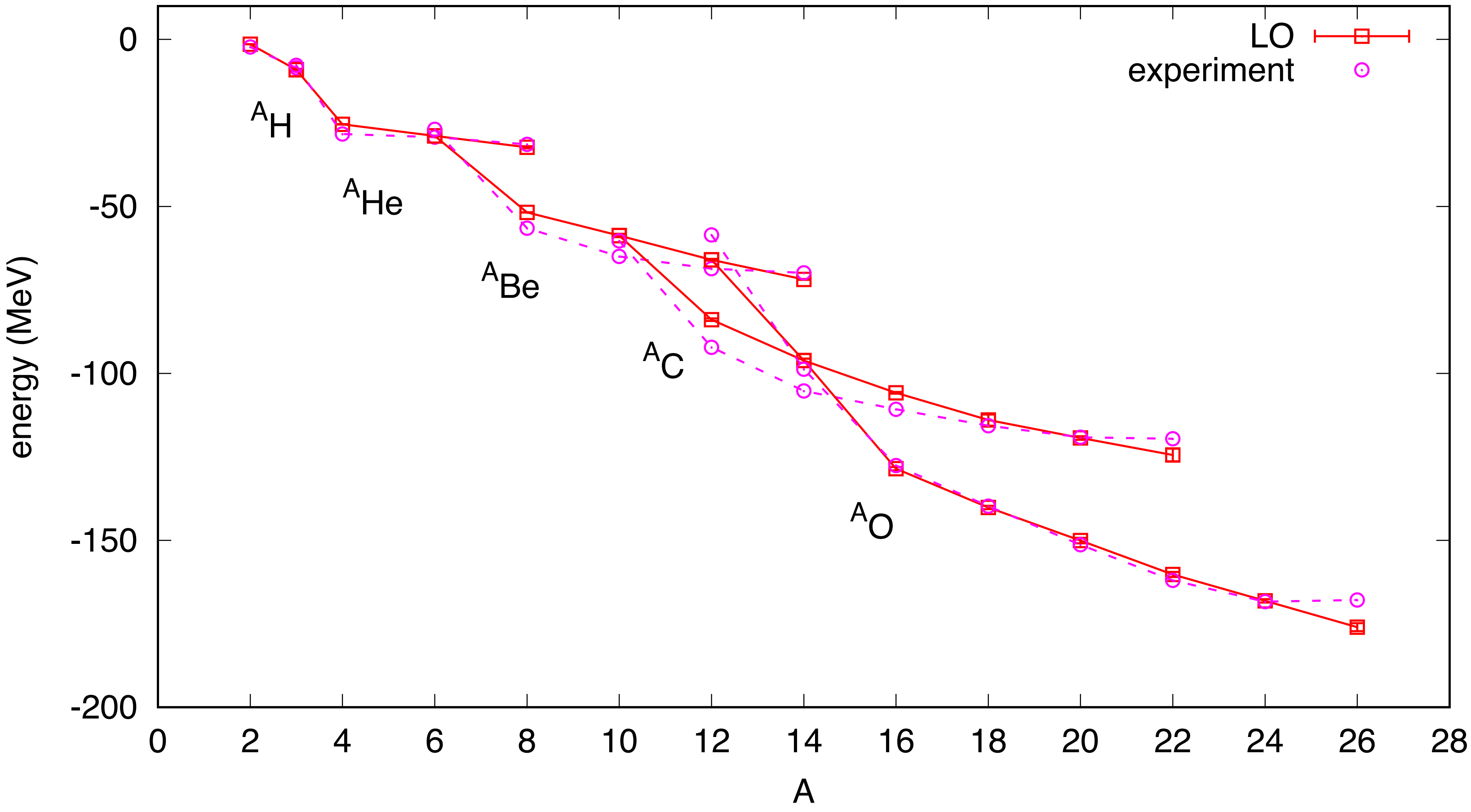
$$a^\dagger_{\rm NL}(\mathbf{n}) = a^\dagger(\mathbf{n}) + s_{\rm NL} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')$$



GROUND STATE ENERGIES

- Fit parameters to average NN S-wave scattering length and effective range and α-α S-wave scattering length

$\rightarrow$ predict g.s. energies of H, He, Be, C and O isotopes $\rightarrow$ quite accurate (LO)



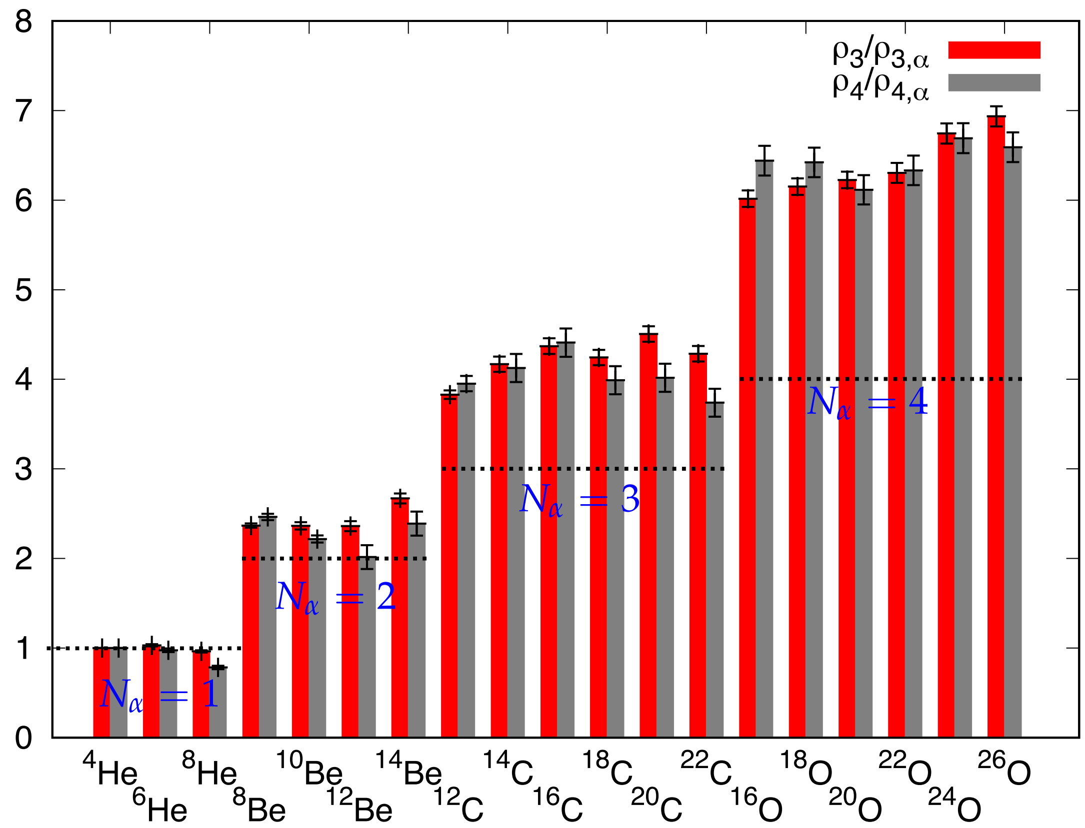
PROBING NUCLEAR CLUSTERING

- Local densities on the lattice: $\rho(\mathbf{n})\,,\quad \rho_p(\mathbf{n})\,,\quad \rho_n(\mathbf{n})$
- Probe of alpha clusters: $\rho_4 = \sum_{\mathbf{n}} :\rho^4(\mathbf{n})/4!:$
- Another probe for $Z = N =$ even nuclei: $\rho_3 = \sum_{\mathbf{n}} :\rho^3(\mathbf{n})/3!:$
- ρ_4 couples to the center of the α-cluster while ρ_3 gets contributions from a wider portion of the alpha-particle wave function
- Both ρ_3 and ρ_4 depend on the regulator, a, but not on the nucleus
- The ratios $\rho_3/\rho_{3,\alpha}$ and $\rho_4/\rho_{4,\alpha}$ free of short-distance ambiguities and model-independent
- $\rho_3/\rho_{3,\alpha}$ measures the effective number of alpha-cluster N_α

$\Rightarrow$ Any deviation from $N_\alpha =$ integer measures the entanglement of the α-clusters in a given nucleus

PROBING NUCLEAR CLUSTERING

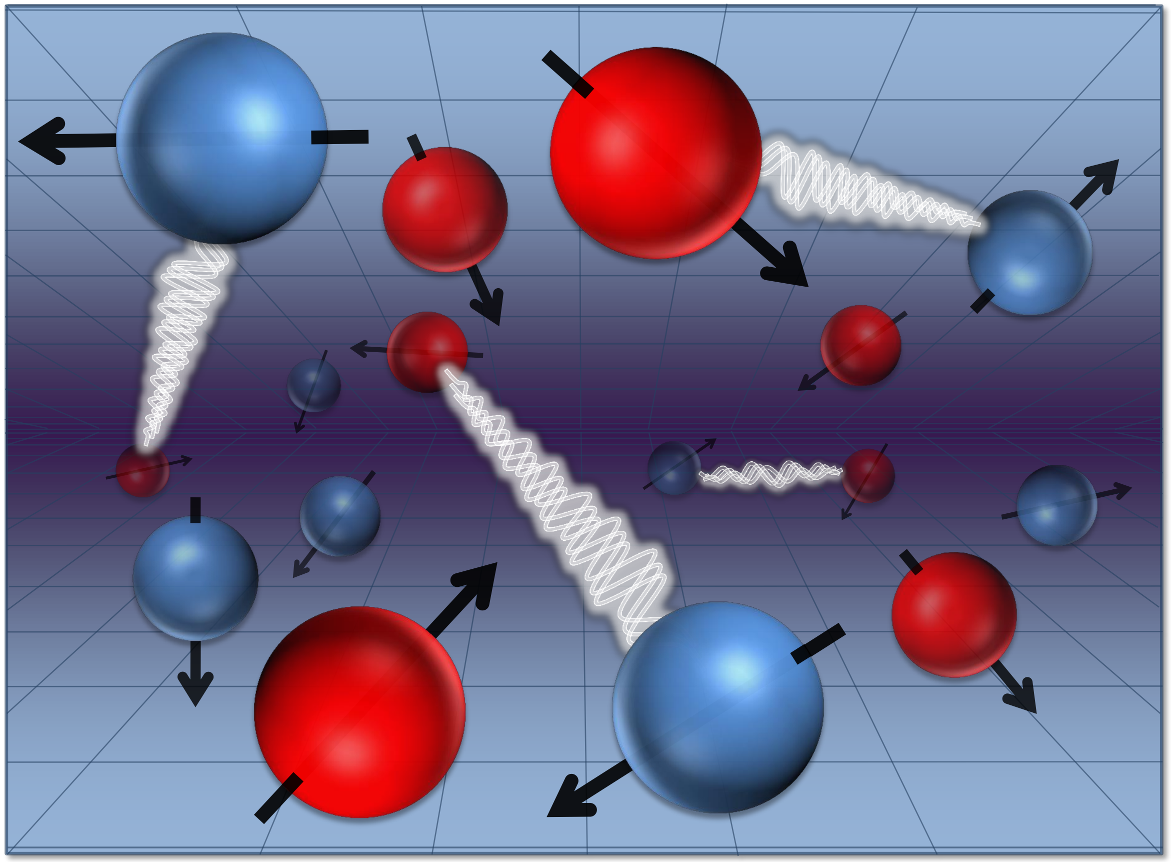
- ρ_3-entanglement of the α-clusters: $$\frac{\Delta_\alpha^{\rho_3}}{N_\alpha} = \frac{\rho_3/\rho_{3,\alpha}}{N_\alpha} - 1$$



Nucleus	^{4,6,8}He	^{8,10,12,14}Be	^{12,14,16,18,20,22}C	^{16,18,20,22,24,26}O
$\Delta_\alpha^{\rho_3}/N_\alpha$	0.00 - 0.03	0.20 - 0.35	0.25 - 0.50	0.50 - 0.75

PROBING NUCLEAR CLUSTERING

- The transition from cluster-like states in light systems to nuclear liquid-like states in heavier systems should not be viewed as a simple suppression of multi-nucleon short-distance correlations, but rather as an increasing *entanglement* of the nucleons involved in the multi-nucleon correlations.



PINHOLE ALGORITHM

- AFQMC calculations involve states that are superpositions of many different center-of-mass positions

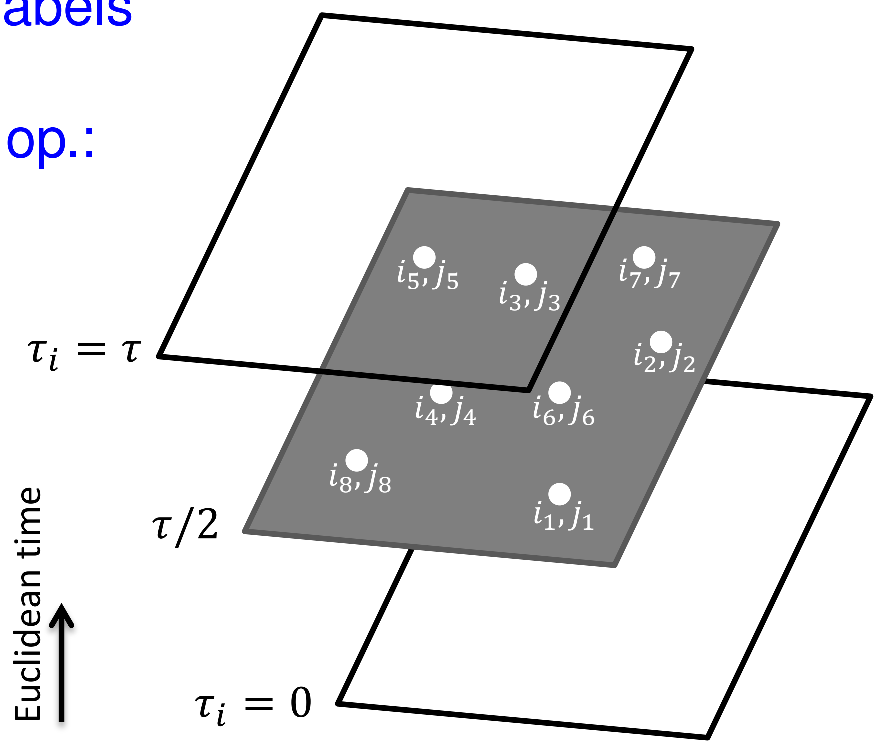
$\longrightarrow$ density distributions of nucleons can not be computed directly

- Insert a screen with pinholes with spin & isospin labels that allows nucleons with corresponding spin & isospin to pass = insertion of the A-body density op.:

$$\rho_{i_1,j_1,\cdots i_A,j_A}(\mathbf{n}_1,\cdots\mathbf{n}_A) = :\rho_{i_1,j_1}(\mathbf{n}_1)\cdots\rho_{i_A,j_A}(\mathbf{n}_A):$$

- MC sampling of the amplitude:

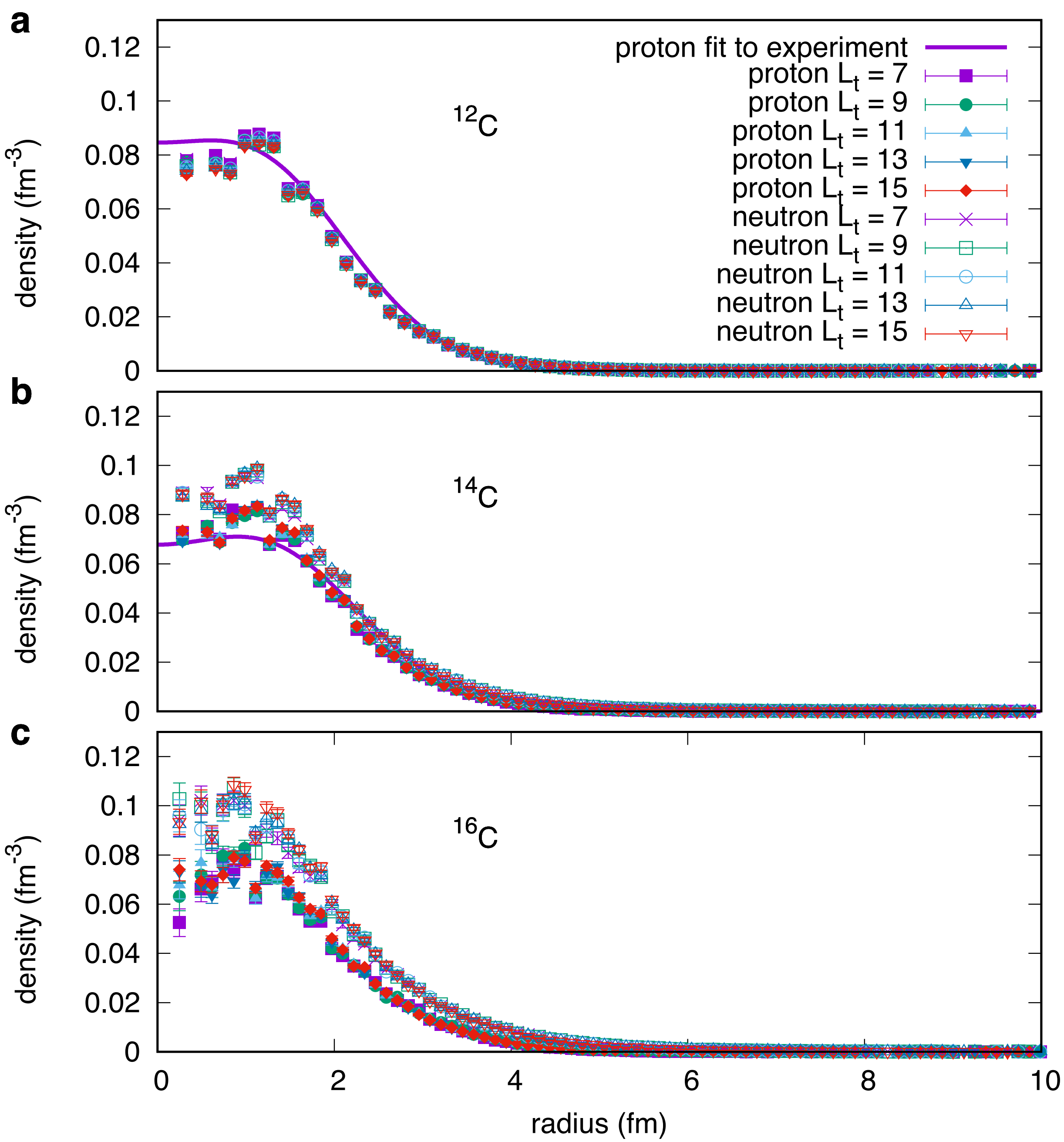
$$A_{i_1,j_1,\cdots i_A,j_A}(\mathbf{n}_1,\ldots,\mathbf{n}_A,L_t) = \langle\psi(\tau/2)|\rho_{i_1,j_1,\cdots i_A,j_A}(\mathbf{n}_1,\ldots,\mathbf{n}_A)|\psi(\tau/2)\rangle$$



- Allows to measure proton and neutron distributions

- Resolution scale $\sim a/A$ as cm position $\mathbf{r}_{\rm cm}$ is an integer $\mathbf{n}_{\rm cm}$ times a/A

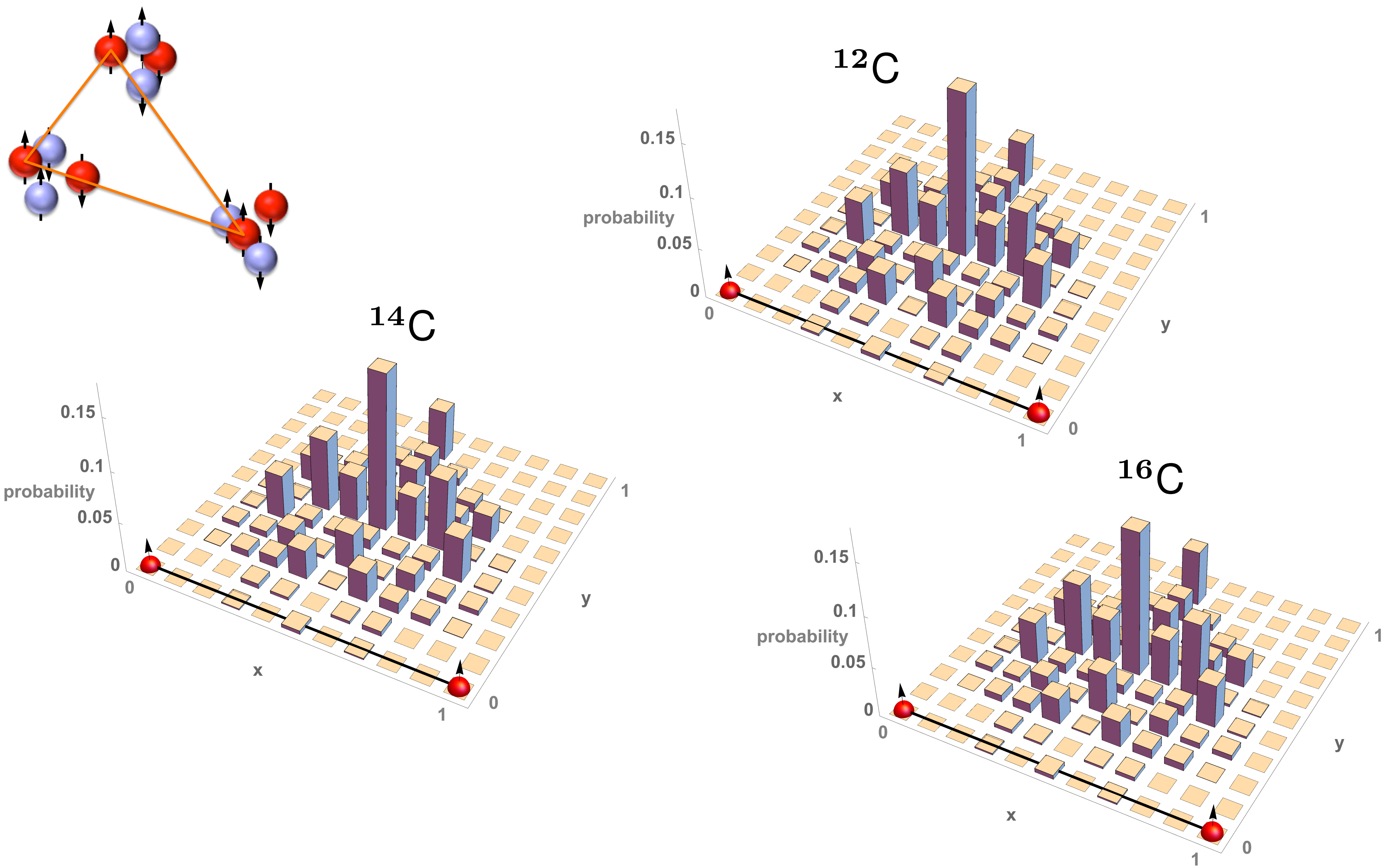
PROTON and NEUTRON DENSITIES in CARBON



- open symbols: neutron
- closed symbols: proton
- proton size accounted for
- asymptotic properties of the distributions from the volume dependence of N-body bound states
  König, Lee, [arXiv:1701.00279]
- consistent with data
- fit to data from
  Kline et al., NPA209 (1973) 381

ALPHA CLUSTER GEOMETRY

- Measuring the three spin-up protons by considering triangular shapes

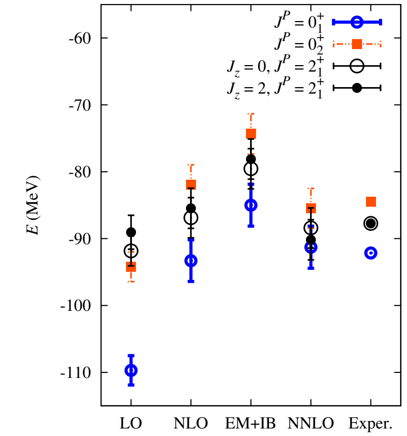


Fine-tunings and the multiverse

UGM, Sci. Bull. **60** (2015) no.1, 43-54

The RELEVANT QUESTION

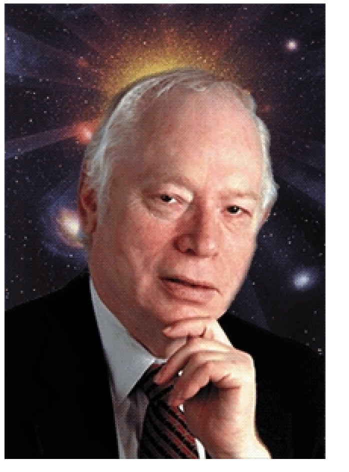
Date: Sat, 25 Dec 2010 20:03:42 -0600
From: Steven Weinberg ⟨weinberg@zippy.ph.utexas.edu⟩
To: Ulf-G. Meissner ⟨meissner@hiskp.uni-bonn.de⟩
Subject: Re: Hoyle state in 12C



Dear Professor Meissner,

Thanks for the colorful graph. It makes a nice Christmas card. But I have a detailed question. Suppose you calculate not only the energy of the Hoyle state in C12, but also of the ground states of He4 and Be8. How sensitive is the result that the energy of the Hoyle state is near the sum of the rest energies of He4 and Be8 to the parameters of the theory? I ask because I suspect that for a pretty broad range of parameters, the Hoyle state can be well represented as a nearly bound state of Be8 and He4.

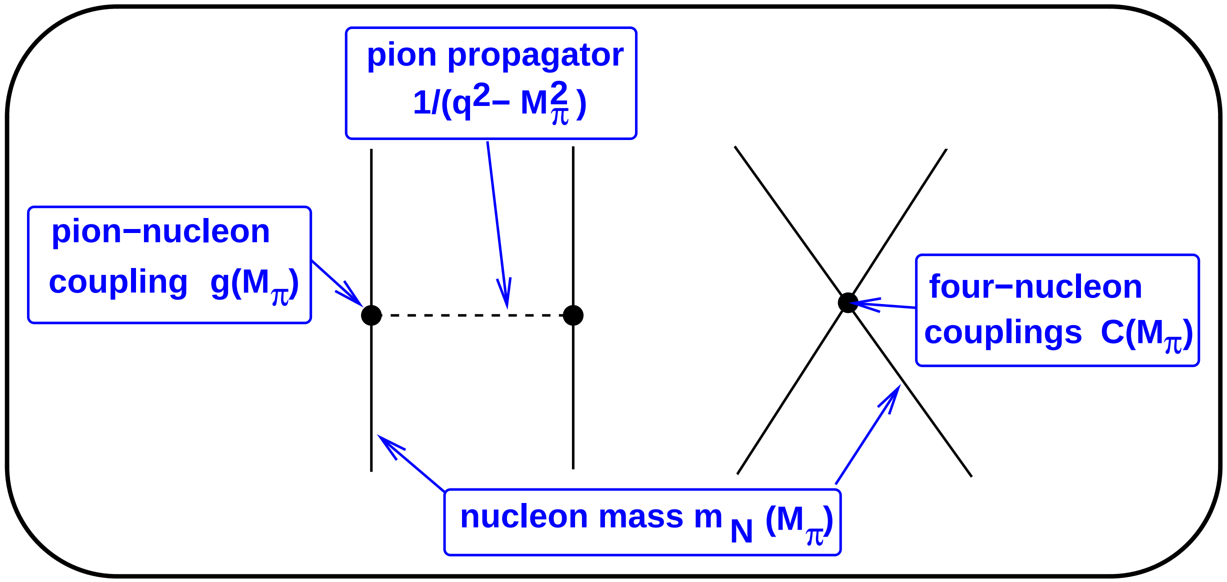
All best,
Steve Weinberg



- How does the Hoyle state move relative to the ^{4}He+^{8}Be threshold, if we change the fundamental parameters of QCD+QED?
- not possible in nature, *but on a high-performance computer!*

NUCLEAR FORCES for VARYING QUARK MASSES

- Nuclear forces: Pion-exchange contributions & short-distance multi-N operators
- graphical representation of the quark mass dependence of the LO potential

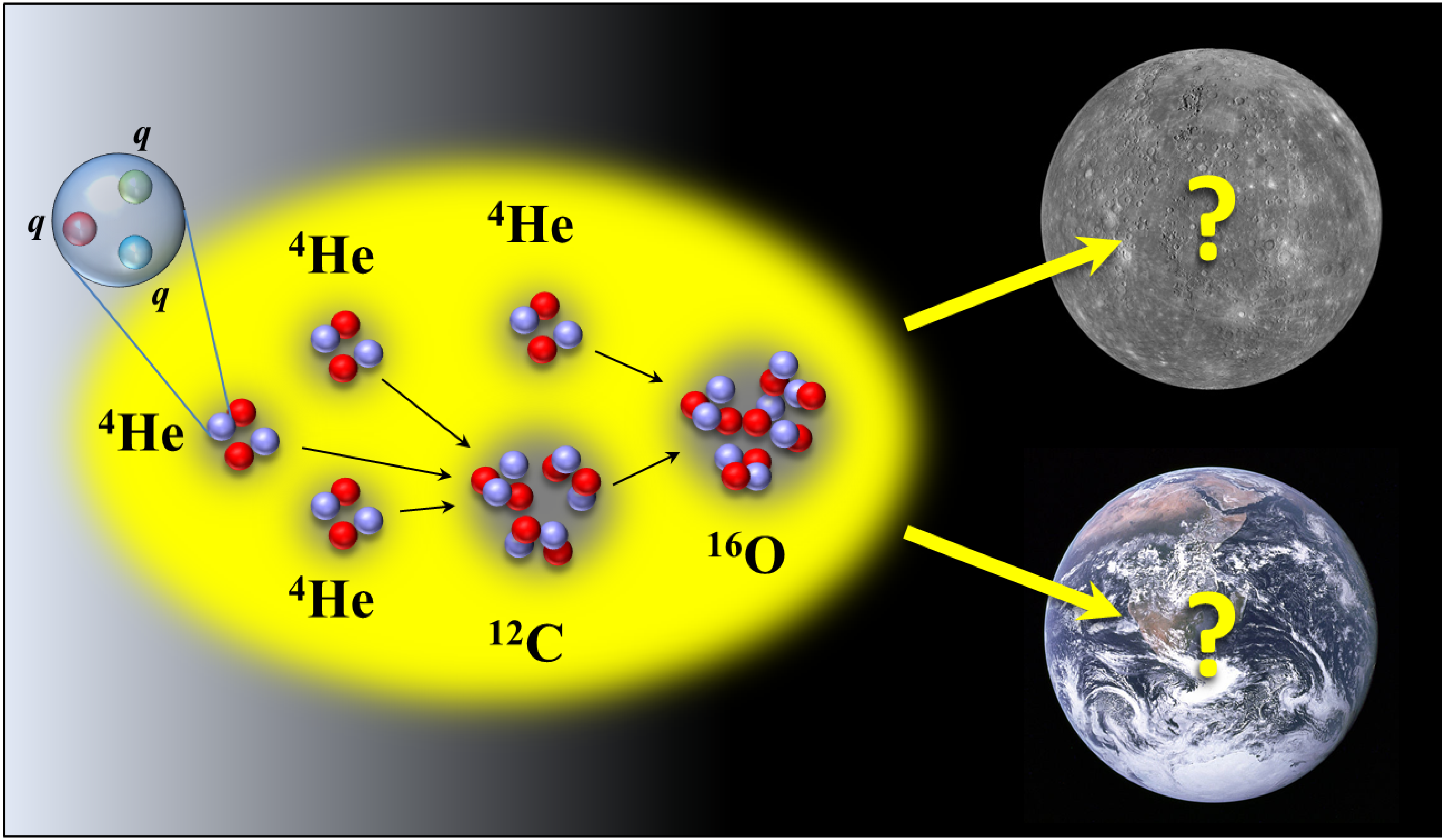


- always use the Gell-Mann–Oakes–Renner relation: $M^2_{\pi^\pm} \sim (m_u + m_d)$
- fulfilled in QCD to better than 94% — Colangelo, Gasser, Leutwyler 2001

$\Rightarrow$ Quark mass dependence of hadron properties from lattice QCD,
contact interaction require modeling $\rightarrow$ challenge to lattice QCD

FINE-TUNING of FUNDAMENTAL PARAMETERS

Fig. courtesy Dean Lee



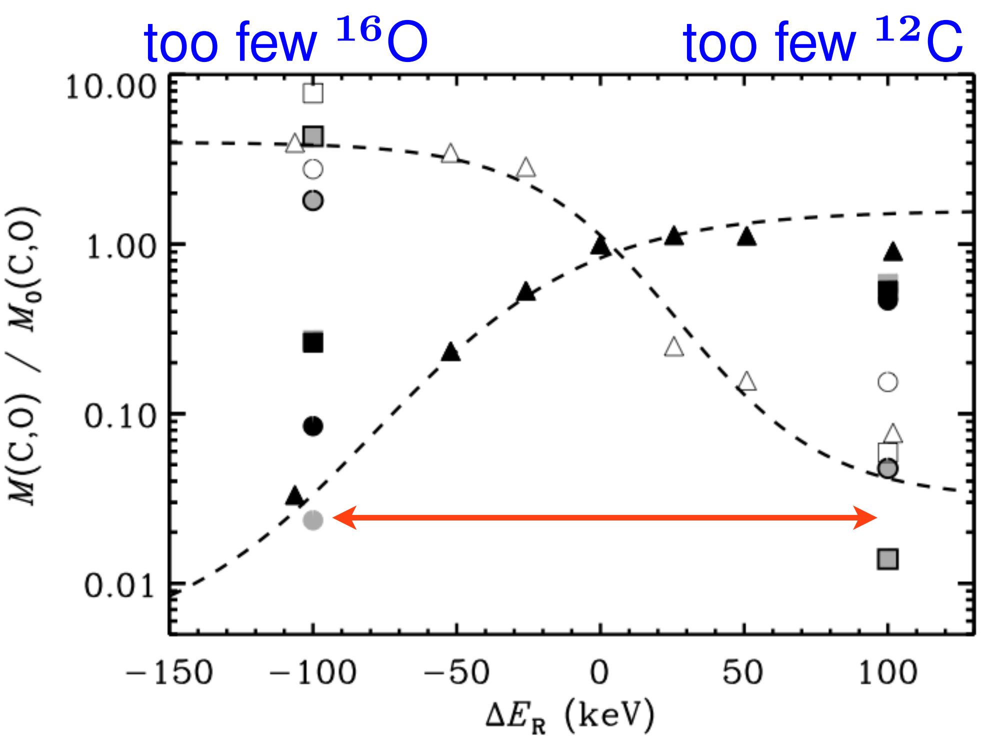
EARLIER STUDIES of the ANTHROPIC PRINCIPLE

- rate of the 3α-process: $r_{3\alpha} \sim \Gamma_\gamma \, \exp\left(-\frac{\Delta E_{h+b}}{kT}\right)$

$$\Delta E_{h+b} = E^\star_{12} - 3E_\alpha = 379.47(18) \text{ keV}$$

- how much can ΔE_{h+b} be changed so that there is still enough ^{12}C and ^{16}O?

$\Rightarrow$ $\delta|\Delta E_{h+b}| \lesssim 100$ keV



Oberhummer et al., Science **289** (2000) 88

Csoto et al., Nucl. Phys. A **688** (2001) 560
Schlattl et al., Astrophys. Space Sci. **291** (2004) 27

[Livio et al., Nature **340** (1989) 281]

FINE-TUNING: MONTE-CARLO ANALYSIS

Epelbaum, Krebs, Lähde, Lee, UGM, PRL **110** (2013) 112502

- consider first QCD only $\to$ calculate $\partial\Delta E/\partial M_\pi$

- relevant quantities (energy *differences*)

^{4}He + ^{4}He $\leftrightarrow$ ^{8}Be $\rightsquigarrow$ $$\boxed{\Delta E_b \equiv E_8 - 2E_4}$$

^{4}He + ^{8}Be $\to$ ^{12}C* $\rightsquigarrow$ $$\boxed{\Delta E_h \equiv E_{12}^* - E_8 - E_4}$$

- energy differences depend on parameters of QCD (LO analysis)

$$\boxed{E_i = E_i\left(M_\pi^{\rm OPE}, m_N(M_\pi), g_{\pi N}(M_\pi), C_0(M_\pi), C_I(M_\pi)\right)}$$

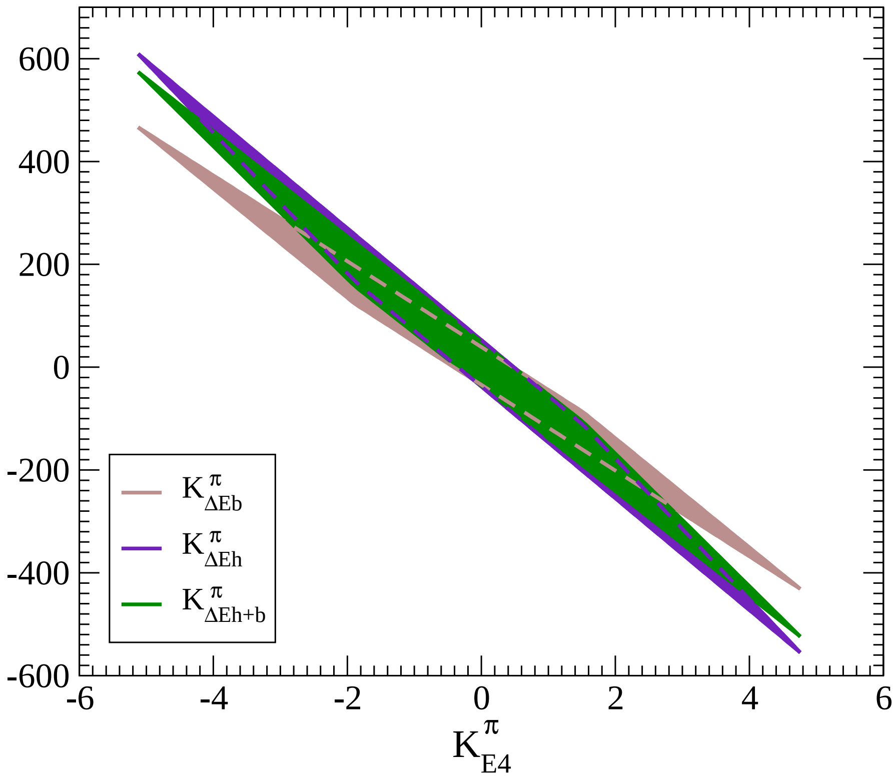
$$g_{\pi N} \equiv g_A/(2F_\pi)$$

- QED in the same manner $\to$ calculate $\partial\Delta E/\partial\alpha_{\rm EM}$

CORRELATIONS

- map $C_{0,I}(M_\pi)$ onto $\bar{A}_{s,t} \equiv \partial a_{s,t}^{-1}/\partial M_\pi\big|_{M_\pi^{\rm phys}}$ [singlet/triplet scatt. length]

- vary the derivatives $\bar{A}_{s,t} \equiv \partial a_{s,t}^{-1}/\partial M_\pi\big|_{M_\pi^{\rm phys}}$ within $-1, \ldots, +1$:



$$\Delta E_b = E(^8\mathrm{Be}) - 2E(^4\mathrm{He})$$

$$\Delta E_h = E(^{12}\mathrm{C}^*) - E(^8\mathrm{Be}) - E(^4\mathrm{He})$$

$$\Delta E_{h+b} = E(^{12}\mathrm{C}^*) - 3E(^4\mathrm{He})$$

$$\boxed{\frac{\partial O_H}{\partial M_\pi} = K_H^\pi \frac{O_H}{M_\pi}}$$

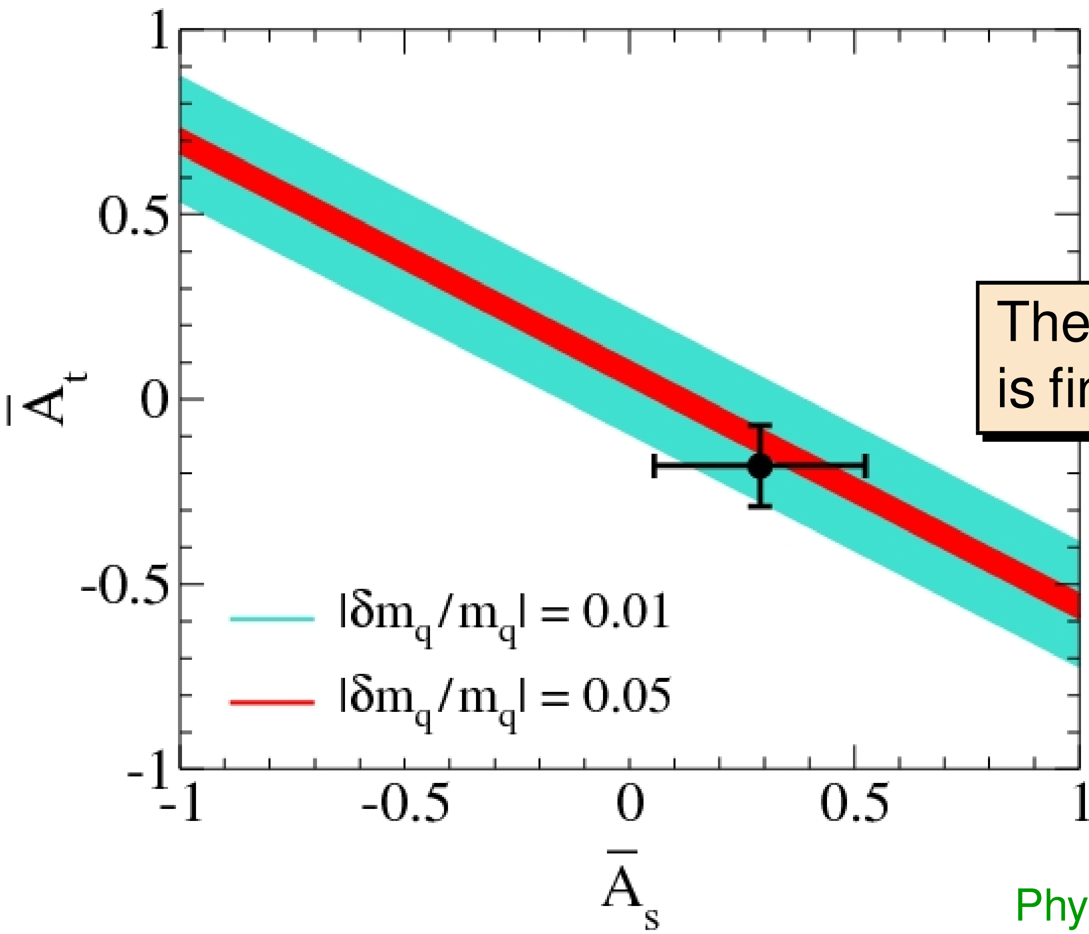
- all fine-tunings in the triple-alpha process are *correlated*, as speculated

Weinberg (2000)

THE END-OF-THE-WORLD PLOT

- $|\delta(\Delta E_{h+b})| < 100$ keV [exp: 387 keV] Oberhummer et al., Science (2000)

$$\rightarrow \left| \left(0.571(14)\,\bar{A}_s + 0.934(11)\,\bar{A}_t - 0.069(6) \right) \frac{\delta m_q}{m_q} \right| < 0.0015$$



$$\bar{A}_{s,t} \equiv \left. \frac{\partial a_{s,t}^{-1}}{\partial M_\pi} \right|_{M_\pi^{\rm phys}}$$

The light quark mass is fine-tuned to $\simeq 2-3\,\%$

Similarly: $\alpha_{\rm EM}$ is fine-tuned to $\simeq 2.5\%$

Berengut et al., Phys. Rev. **D 87** (2013) 085018

SUMMARY & OUTLOOK

- Chiral EFT for nuclear forces

  $\to$ precise framework for 2N and 3N forces with small uncertainties

  $\to$ can also be formulated at varying strong and em forces

- Nuclear lattice simulations: a new quantum many-body approach

  $\to$ based on the successful continuum nuclear chiral EFT

  $\to$ a number of intriguing results already obtained

  $\to$ clustering emerges naturally, α-cluster nuclei

  '$\to$ fine-tuning in nuclear reactions can be studied

- Various bridges to lattice QCD studies need to be explored

- Many open issues can now be addressed in a truly quantitative manner

  $\to$ the "holy grail" of nuclear astrophysics ^{4}He+$^{12}\mathrm{C} \to ^{16}\mathrm{O} + \gamma$

Fowler (1983)

SPARES

EFFECTIVE FIELD THEORY in a NUTSHELL

Weinberg, Gasser, Leutwyler, ...

- Rules to construct an EFT:

> - *scale separation* – what is low, what is high?
> - *active degrees of freedom* – what are the building blocks?
> - *symmetries* – how are the interactions constrained by symmetries?
> - *power counting* – how to organize the expansion in low over high?

- QCD with light quarks (up, down):

low scale $\sim M_\pi \ll$ high scale $\sim M_\rho$

DOFs: pions = Goldstone bosons, nucleons, ...

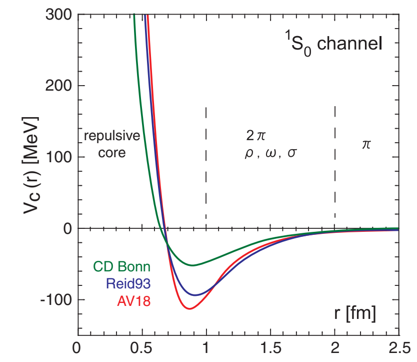
broken chiral symmetry, PCT, Lorentz, ...

Amp $\sim q^\nu$, $\nu = 4 - N + 2(L - C) + \sum_i V_i \Delta_i$

CHIRAL EFT for FEW-NUCLEON SYSTEMS

Gasser, Leutwyler, Weinberg, van Kolck, Epelbaum, Bernard, Kaiser, UGM, . . .

- Scales in nuclear physics:



Natural: $\lambda_\pi = 1/M_\pi \simeq 1.5$ fm (Yukawa 1935)

Unnatural: $|a_{np}(^1S_0)| = 23.8\,\text{fm}\ ,\ a_{np}(^3S_1) = 5.4\,\text{fm} \gg 1/M_\pi$

- this can be analyzed in a suitable EFT based on

$$\mathcal{L}_{\rm QCD} \to \mathcal{L}_{\rm EFF} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \mathcal{L}_{NN} + \ldots$$

- pion and pion-nucleon sectors are perturbative in Q/Λ_χ → chiral perturbation th'y
- $\mathcal{L}_{NN}$ collects short-distance contact terms, to be fitted
- NN interaction requires non-perturbative resummation

→ chirally expand $\mathrm{V_{NN(N)}}$, use in regularized Schrödinger equation

NUCLEAR FORCES: OPEN ENDS

- Why is there this hierarchy $\boxed{V_{2N} \gg V_{3N} \gg V_{4N}}$?
- Gauge and chiral symmetries difficult to include (meson-exchange currents)

Brown, Riska, Gari, . . .

- Connection to QCD ?

most models have one-pion-exchange, but not necessarily respect chiral symmetry

some models have two-pion exchange reconstructed via dispersion relations from $\pi N \to \pi N$

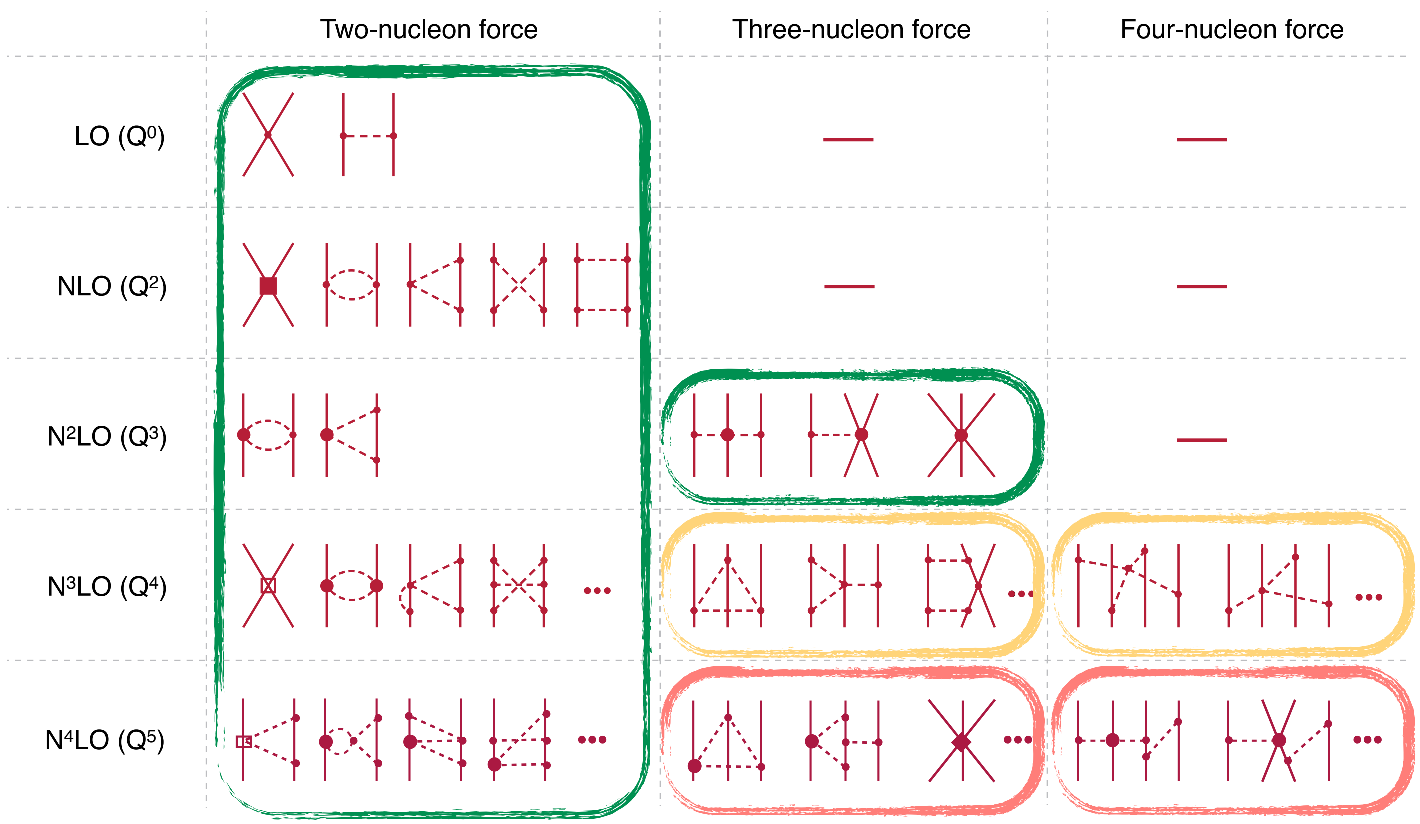
⇒ We want an approach that

— is linked to QCD via its symmetries
— allows for systematic calc's with a controlled theoretical error
— explains the observed hierarchy of the nuclear forces
— matches nucleon structure to nuclear dynamics
— allows for a lattice formulation / chiral extrapolations
— puts nuclear physics on a sound basis

NUCLEAR FORCES in CHIRAL NUCLEAR EFT

- expansion of the potential in powers of Q [small parameter]: $\{p/\Lambda_b, M_\pi/\Lambda_b\}$
- explains observed hierarchy of the nuclear forces
- extremely successful in few-nucleon systems

Epelbaum, Hammer, UGM, Rev. Mod. Phys. **81** (2009) 1773

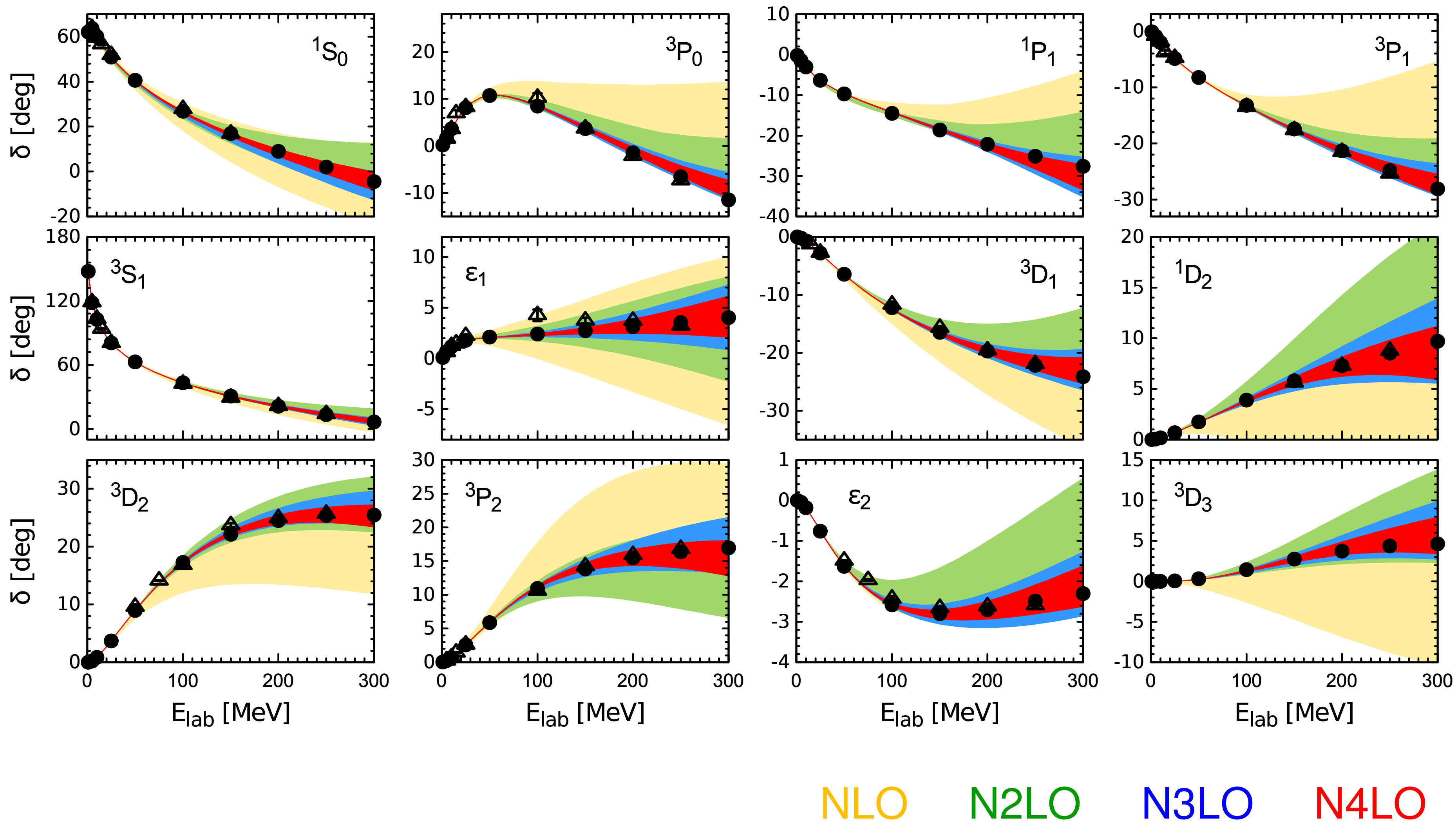


worked out and applied   worked out and to be applied   calculations in progress

PHASE SHIFTS at N4LO

⇒ Precision phase shifts with small uncertainties up to $E_{\rm lab} = 300$ MeV

Epelbaum, Krebs, UGM, Phys. Rev. Lett. **115** (2015) 122301

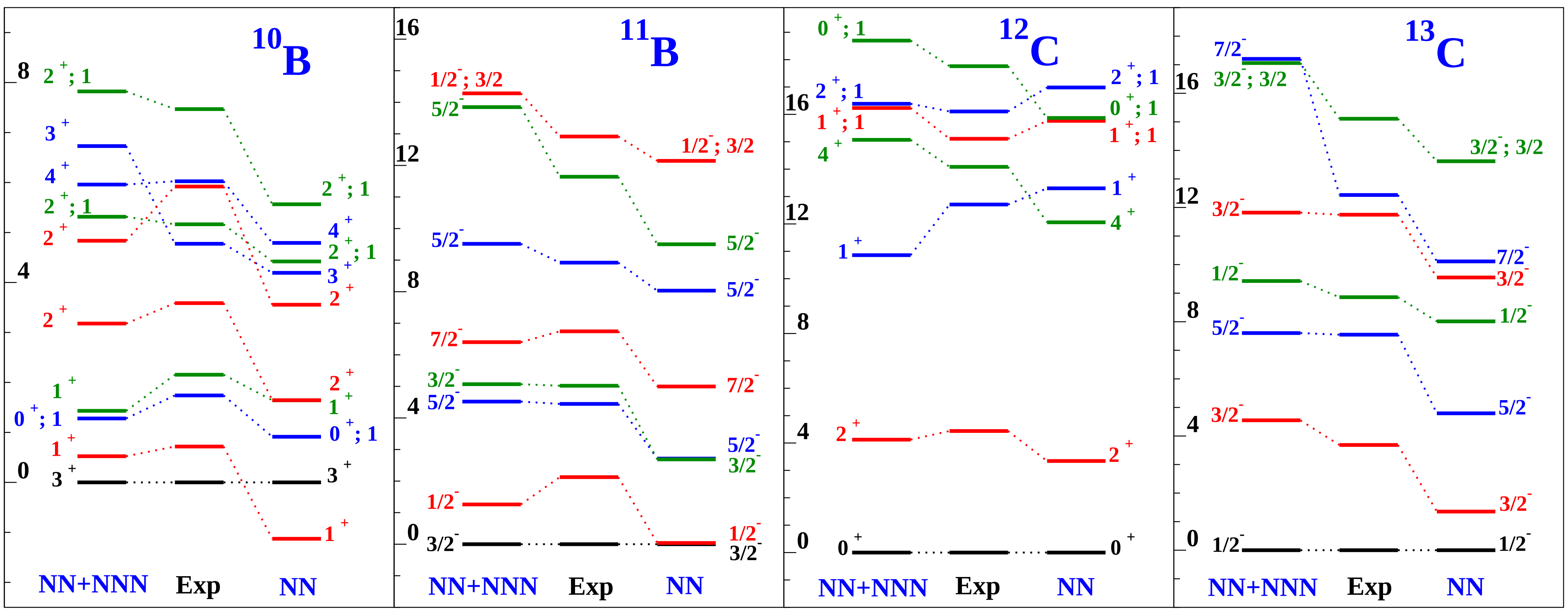


NO-CORE-SHELL MODEL: p-SHELL NUCLEI

- No-core-shell-model calculation — Navratil *et al.*, Phys. Rev. Lett. **99**, 042501 (2007)

- NN interaction at N^3LO and NNN interaction at N^2LO

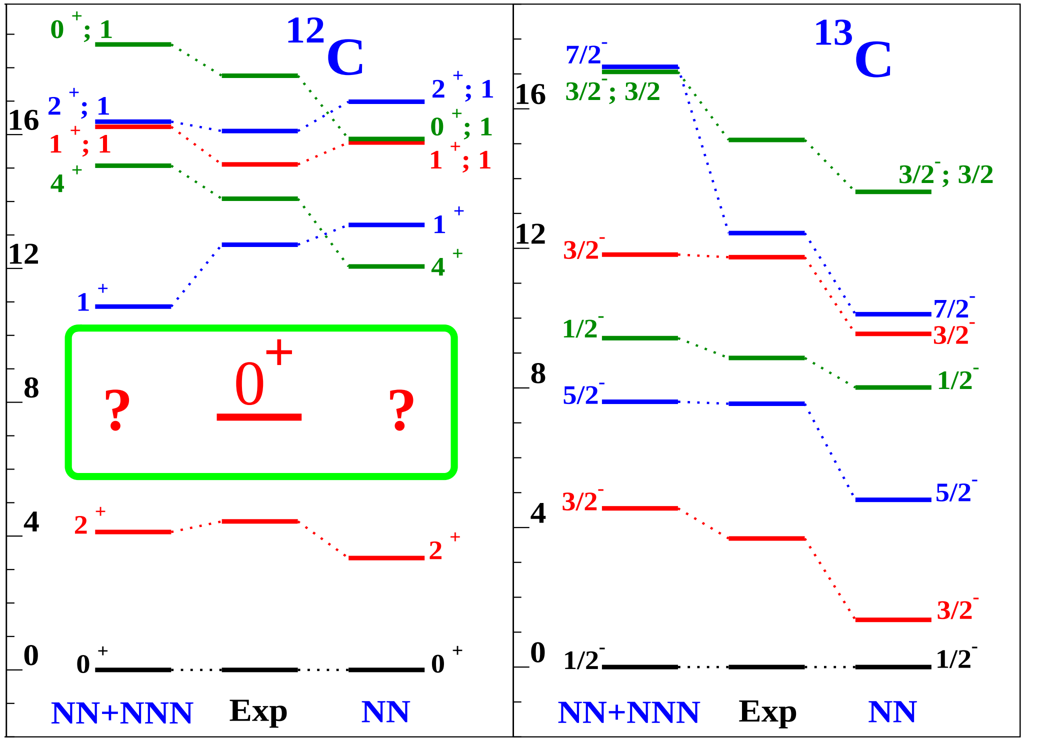
- Fix $D\&E$ from BE of ^{3}H and level structure of ^{4}He, ^{6}Li, ^{10,11}B and ^{12,13}C



MODERN MANY-BODY THEORY and the HOYLE STATE

- one of the most sophisticated many-body theories (No-Core-Shell-Model)
- excellent description of p-shell nuclei from ^{6}Li to ^{13}C

P. Navratil et al., Phys. Rev. Lett. **99** (2007) 042501 + updates



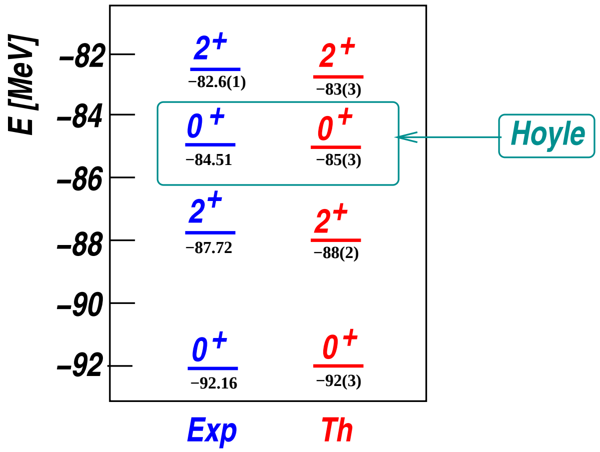
⇒ NO signal of the Hoyle state (i.g. α-cluster states)

⇒ must develop a better method

RESULTS from LATTICE NUCLEAR EFT

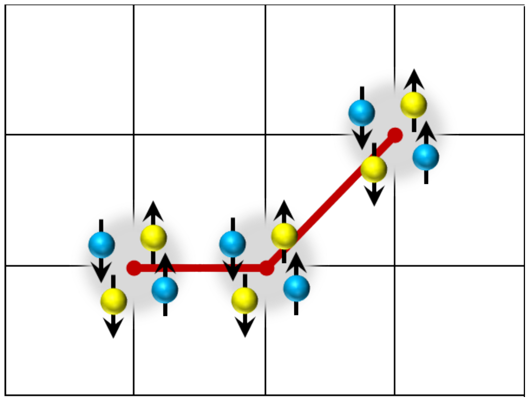
- Hoyle state in ^{12}C

PRL **106** (2011)



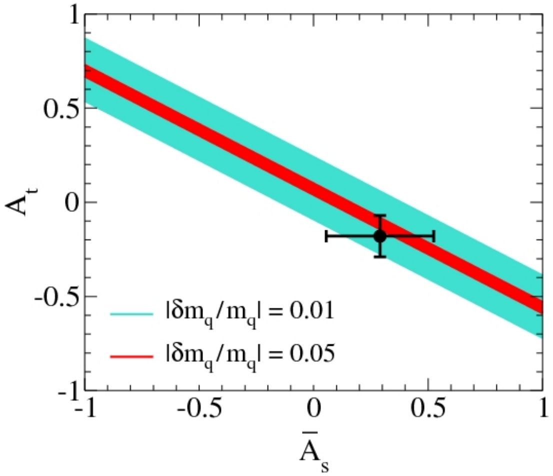
- Structure of the Hoyle state

PRL **109** (2012)



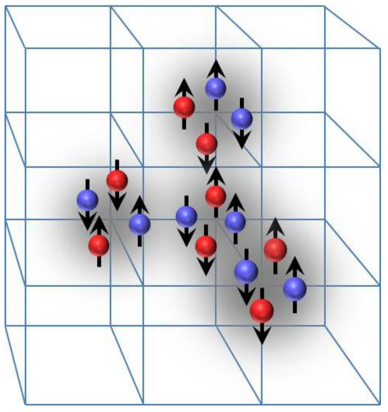
- Fate of carbon-based life

PRL **110** (2013), EPJA **49** (2013)



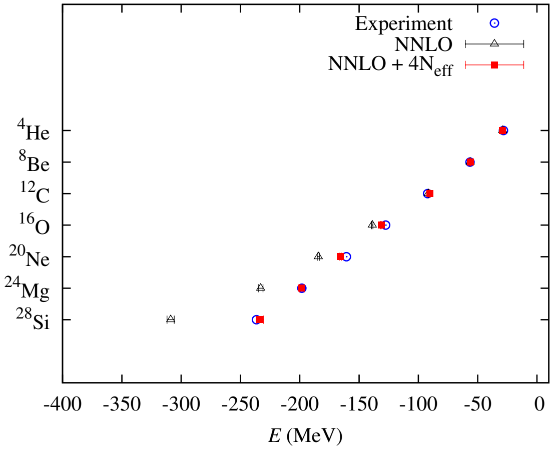
- Spectrum of ^{16}O

PRL **112** (2014)



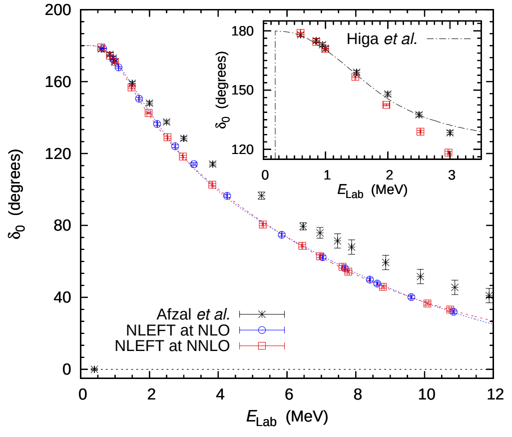
- Going up the α-chain

PLB **732** (2014)



- Ab initio α-α scattering

Nature **528** (2015)

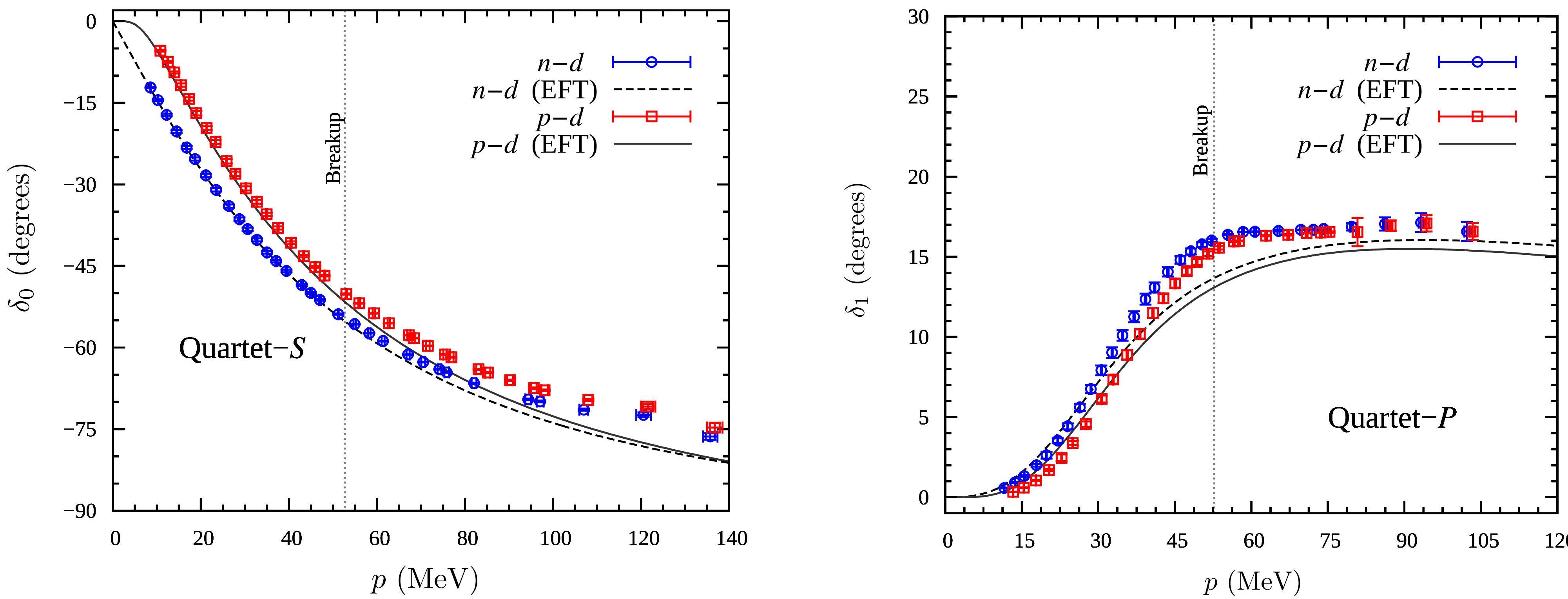


ANOTHER TEST: NUCLEON–DEUTERON SCATTERING

Elhatisari, Lee, UGM, Rupak, Eur. Phys. J. A **52** (2016) 174

- Use improved methods (cluster states projected on sph. harmonics, etc.) & algorithmic improvements
- Precision calculation of proton-deuteron and neutron-deuteron scattering

Pionless EFT: König, Hammer, Gabbiani, Bedaque, Rupak, Griesshammer, van Kolck, 1998-2011



Nuclear binding near a quantum phase transition

Elhatisari, Li, Rokash, Alarcon, Du, Klein, Lu, UGM, Epelbaum,
Krebs, Lähde, Lee, Rupak,
Phys. Rev. Lett. **117** (2016) 132501 [arXiv:1602.04539]

Editors' suggestion, featured in Physics viewpoint: D.J. Dean, Physics 9 (2016) 106

GENERAL CONSIDERATIONS

- *Ab initio* chiral EFT is an excellent theoretical framework

- not guaranteed to work well with increasing A

$\rightarrow$ possible sources of problems:

higher-body forces, higher orders, cutoff dependence, ...

- very many ways of formulating chiral EFT at any given order (smearing etc.)

$\rightarrow$ use not only NN scattering and light nuclei BEs
but also light nucleus-nucleus scattering data
to pin down the pertinent interactions

$\rightarrow$ troublesome corrections might be small

$\rightarrow$ investigate these issues using two seemingly equivalent interactions
[not a precision study!]

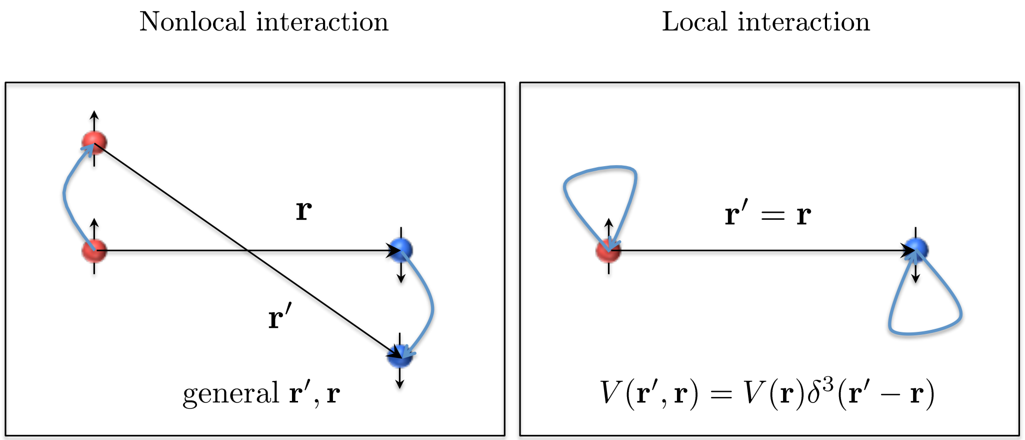
LOCAL and NON–LOCAL INTERACTIONS

- General potential: $V(\vec{r}, \vec{r}\,')$

- Two types of interactions:

  local: $\vec{r} = \vec{r}\,'$

  non-local: $\vec{r} \neq \vec{r}\,'$

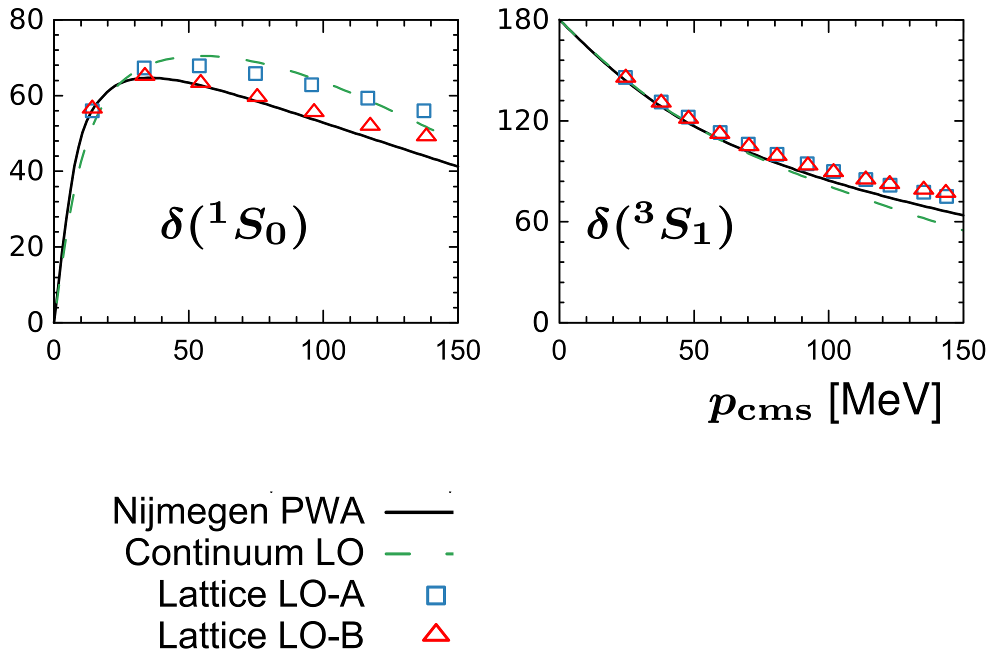


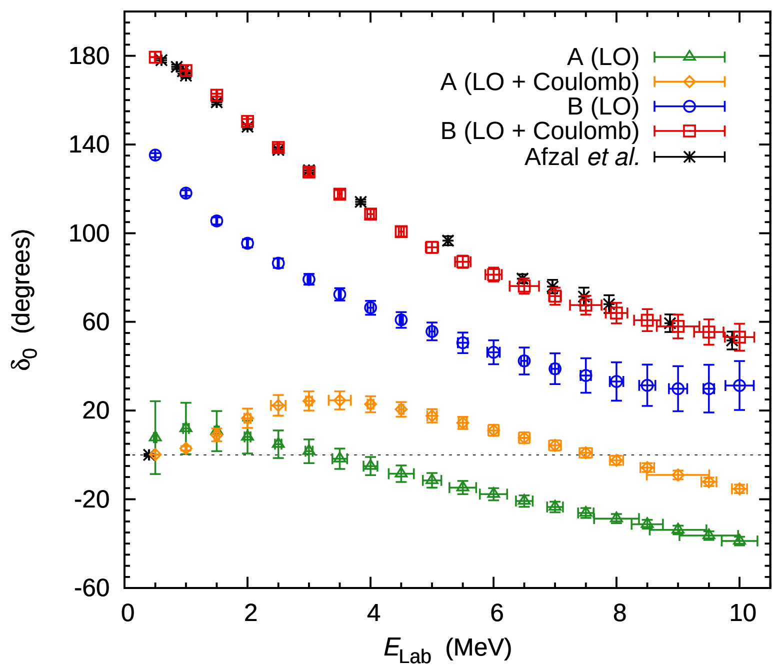
- Taylor two very different interactions:

Interaction A at LO (+ Coulomb)	Interaction B at LO (+ Coulomb)
Non-local short-range interactions	Non-local short-range interactions
+ One-pion exchange interaction	+ Local short-range interactions
(+ Coulomb interaction)	+ One-pion exchange interaction
	(+ Coulomb interaction)
$\rightarrow$ tuned to NN phase shifts	$\rightarrow$ tuned to NN + α-α phase shifts

NN and ALPHA–ALPHA PHASE SHIFTS

- Both interactions very similar for NN but **not** for α-α phase shifts:



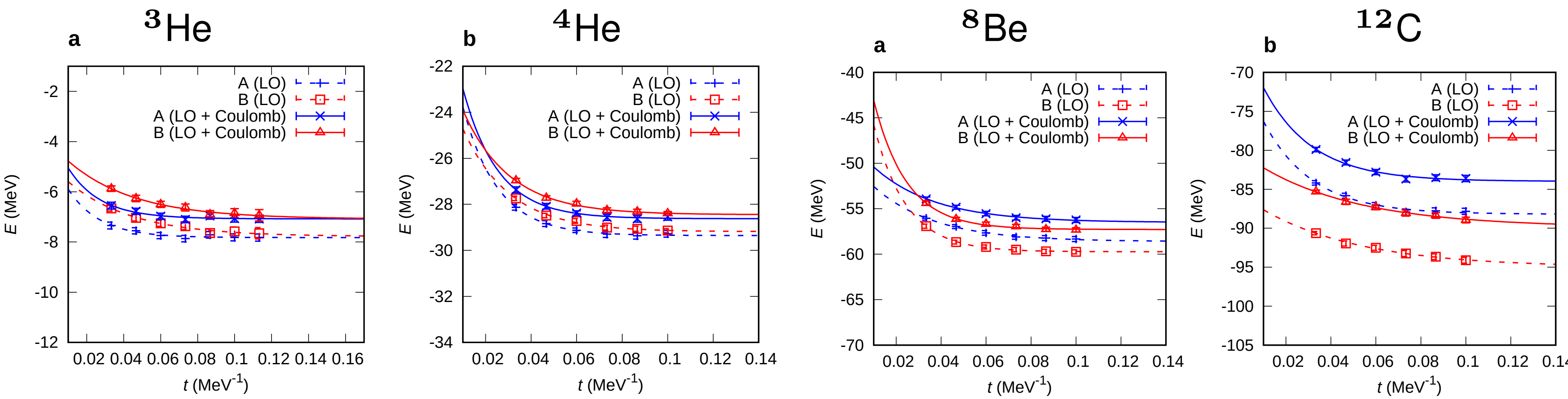


$\rightarrow$ Interaction A fails, interaction B fitted

$\hookrightarrow$ consequences for nuclei?

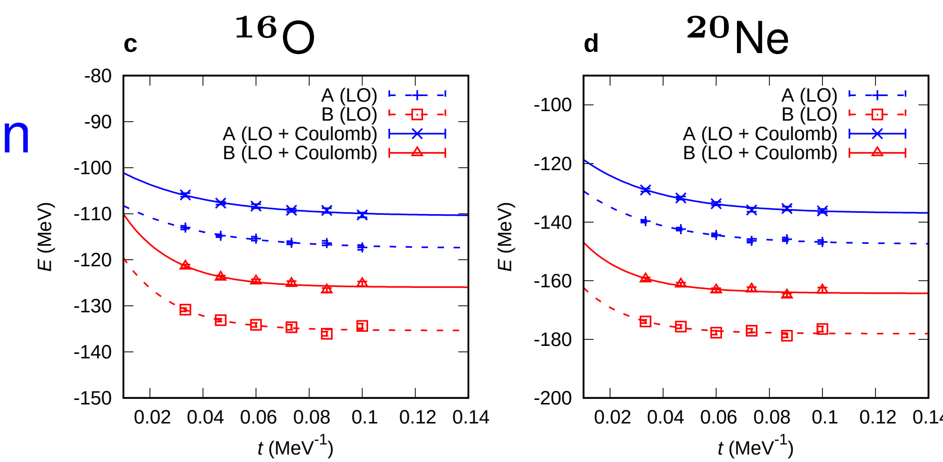
GROUND STATE ENERGIES I

- Ground state energies for alpha-type nuclei plus ^{3}He:



- small errors, excellent thermalization

⇒ Both interactions significantly reduce the Monte Carlo sign oscillations (original motivation)



GROUND STATE ENERGIES I

- Ground state energies for alpha-type nuclei (in MeV):

	A (LO)	A (LO+C.)	B (LO)	B (LO+C.)	Exp.
^{4}He	$-29.4(4)$	$-28.6(4)$	$-29.2(1)$	$-28.5(1)$	-28.3
^{8}Be	$-58.6(1)$	$-56.5(1)$	$-59.7(6)$	$-57.3(7)$	-56.6
^{12}C	$-88.2(3)$	$-84.0(3)$	$-95.0(5)$	$-89.9(5)$	-92.2
^{16}O	$-117.5(6)$	$-110.5(6)$	$-135.4(7)$	$-126.0(7)$	-127.6
^{20}Ne	$-148(1)$	$-137(1)$	$-178(1)$	$-164(1)$	-160.6

- B (LO+Coulomb) quite close to experiment (within 2% or better)

- A (LO) describes a Bose condensate of particles:

$$E(^8\mathrm{Be})/E(^4\mathrm{He}) = 1.997(6) \qquad E(^{12}\mathrm{C})/E(^4\mathrm{He}) = 3.00(1)$$

$$E(^{16}\mathrm{O})/E(^4\mathrm{He}) = 4.00(2) \qquad E(^{20}\mathrm{Ne})/E(^4\mathrm{He}) = 5.03(3)$$

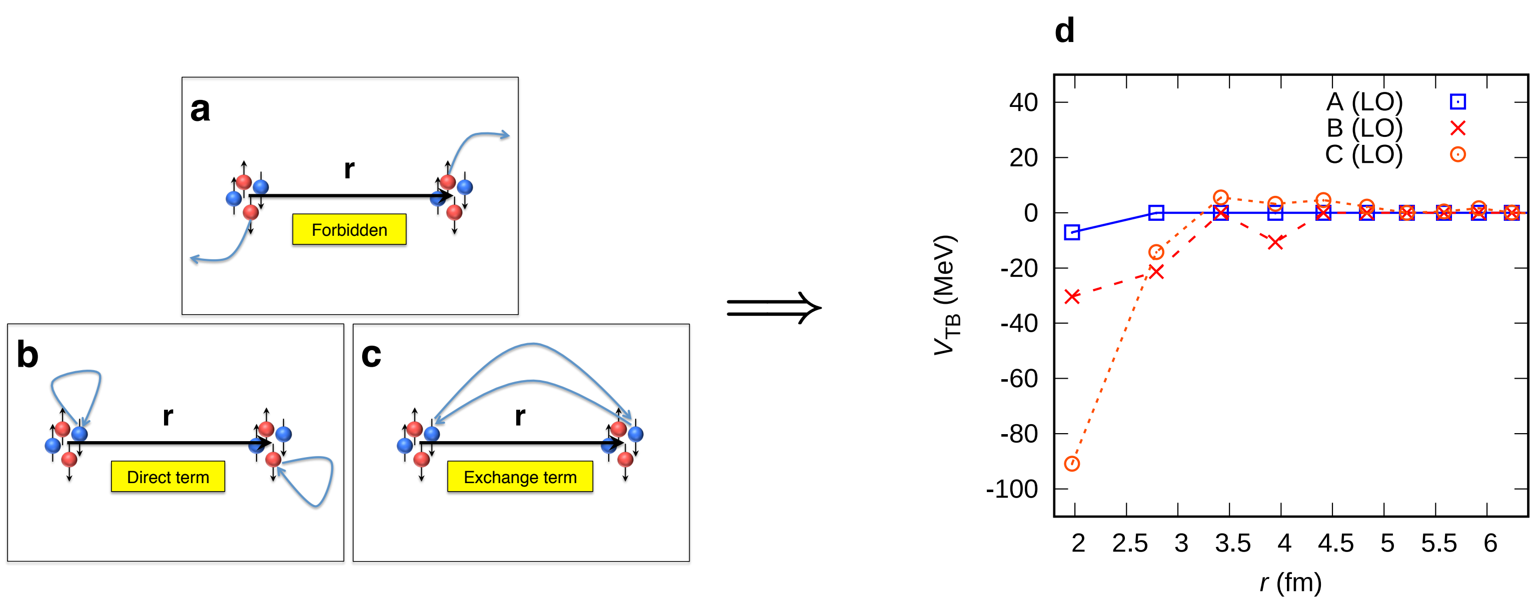
FIRST INSIGHT

- Interaction B was tuned to the nucleon-nucleon phase shifts, the deuteron binding energy, and the S-wave α-α phase shift

- Interaction A starts from interaction B, but *all* local short-distance interactions are switched off, then the LECs of the non-local terms are refitted to describe the nucleon-nucleon phase shifts and the deuteron binding energy

$\rightarrow$ The alpha-alpha interaction is sensitive to the degree of locality of the NN int.

$\rightarrow$ Qualitative understanding: tight-binding approximation (eff. α-α int.)



CONSEQUENCES for NUCLEI and NUCLEAR MATTER

- Define a one-parameter family of interactions that interpolates between the interactions A and B:

$$\boxed{V_\lambda = (1-\lambda)\,V_A + \lambda\,V_B}$$

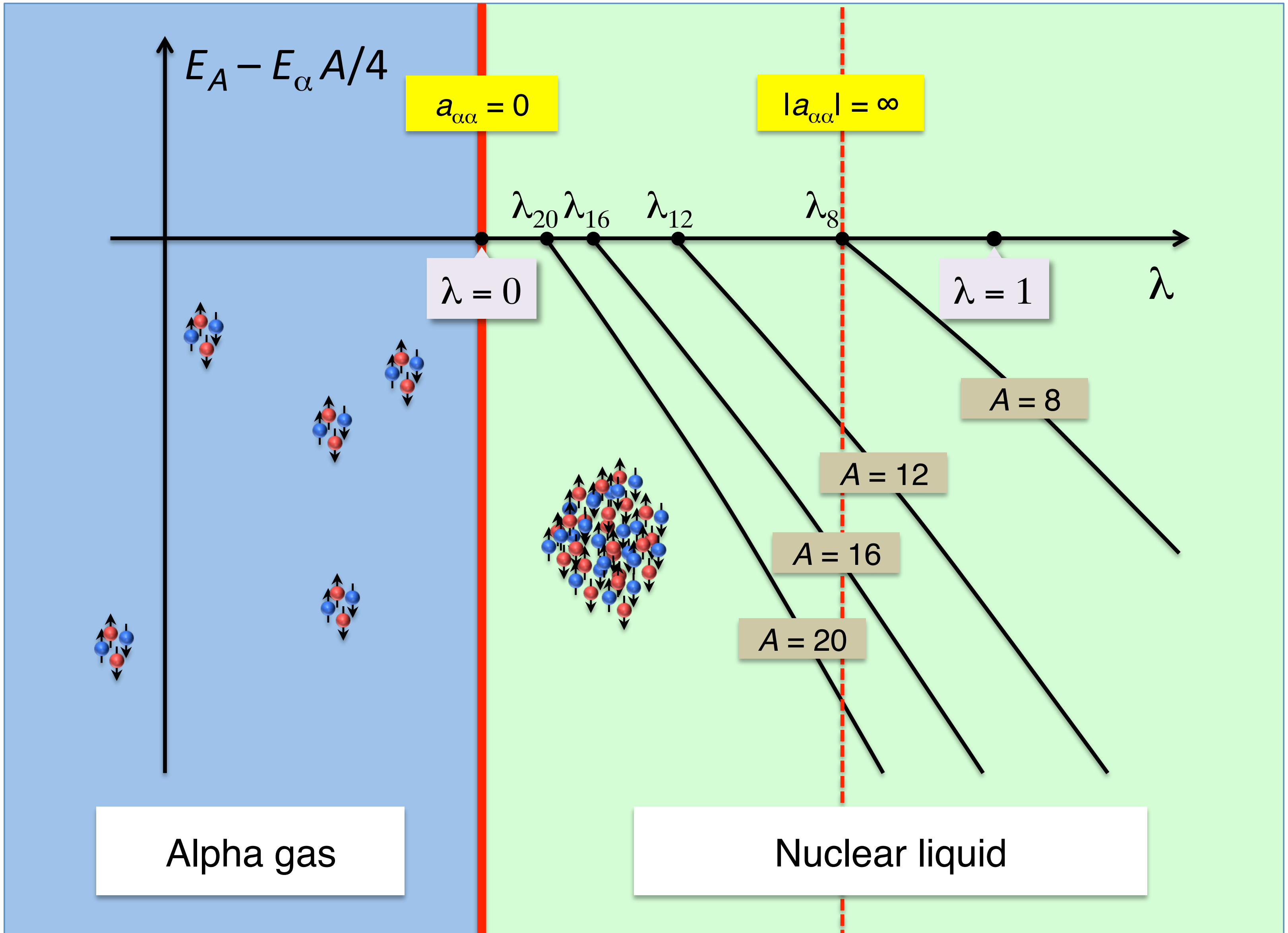
- To discuss the many-body limit, we turn off the Coulomb interaction and explore the zero-temperature phase diagram

- As a function of λ, there is a quantum phase transition at the point where the alpha-alpha scattering length vanishes

  Stoff, Phys. Rev. A **49** (1994) 3824

- The transition is a first-order transition from a Bose-condensed gas of alpha particles to a nuclear liquid

ZERO–TEMPERATURE PHASE DIAGRAM



$$\lambda_8 = 0.7(1)$$
$$\lambda_{12} = 0.3(1)$$
$$\lambda_{16} = 0.2(1)$$
$$\lambda_{20} = 0.2(1)$$
$$\lambda_\infty = 0.0(1)$$

FURTHER CONSEQUENCES

- By adjusting the parameter λ in *ab initio* calculations, one can move the of any α-cluster state up and down to alpha separation thresholds.

$\rightarrow$ This can be used as a new window to view the structure of these exotic nuclear states

- In particular, one can tune the α-α scattering length to infinity!

$\rightarrow$ In the absence of Coulomb interactions, one can thus make contact to **universal Efimov physics**:

for a review, see Braaten, Hammer, Phys. Rept. **428** (2006) 259

Hoyle state of ^{12}C	$\xrightarrow{\lambda}$	Universal Efimov trimer
Second 0^+ of ^{16}O	$\xrightarrow{\lambda}$	Universal Efimov tetramer

SCATTERING CLUSTER WAVE FUNCTIONS

- During Euclidean time interval τ_ε, each cluster undergoes spatial diffusion:

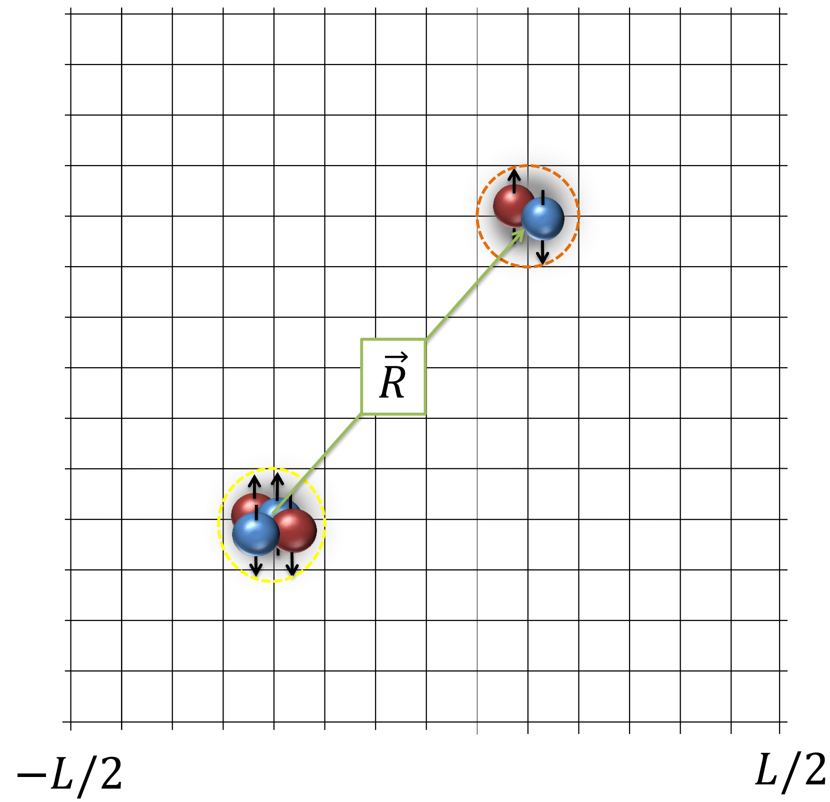
$$d_{\varepsilon,i} = \sqrt{\tau_\varepsilon / M_i}$$

- Only non-overlapping clusters if

$$|\vec{R}| \gg d_{\varepsilon,i} \quad \Rightarrow \quad |\vec{R}\rangle_{\tau_\varepsilon}$$

- Defines asymptotic region, where the amount of overlap between clusters is less than ε

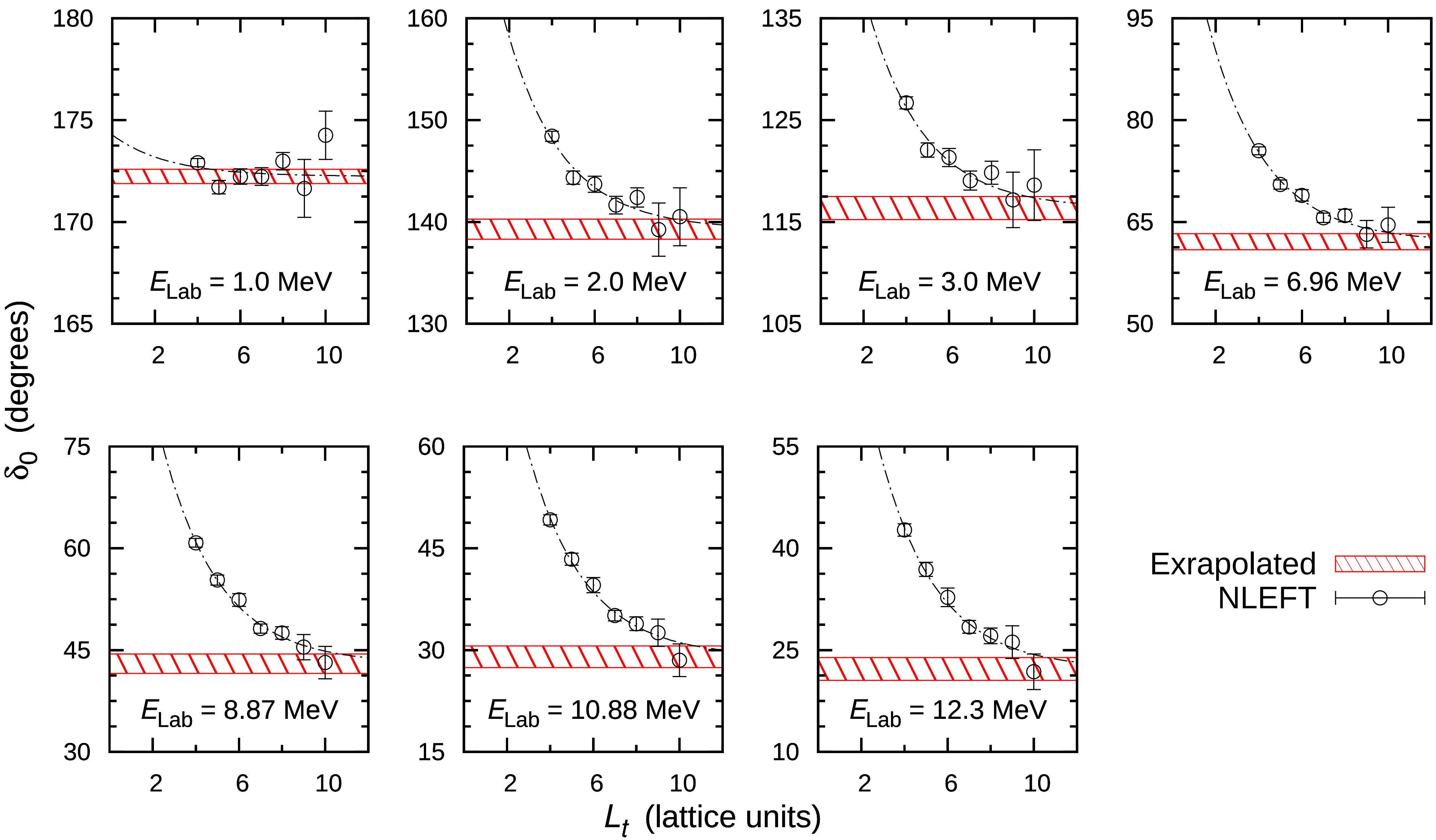
$$|\vec{R}| > R_\varepsilon$$



$\Rightarrow$ In the asymptotic region we can describe the system in terms of an effective cluster Hamiltonian (the free lattice Hamiltonian for two clusters) plus infinite-range interactions (like the Coulomb int.)

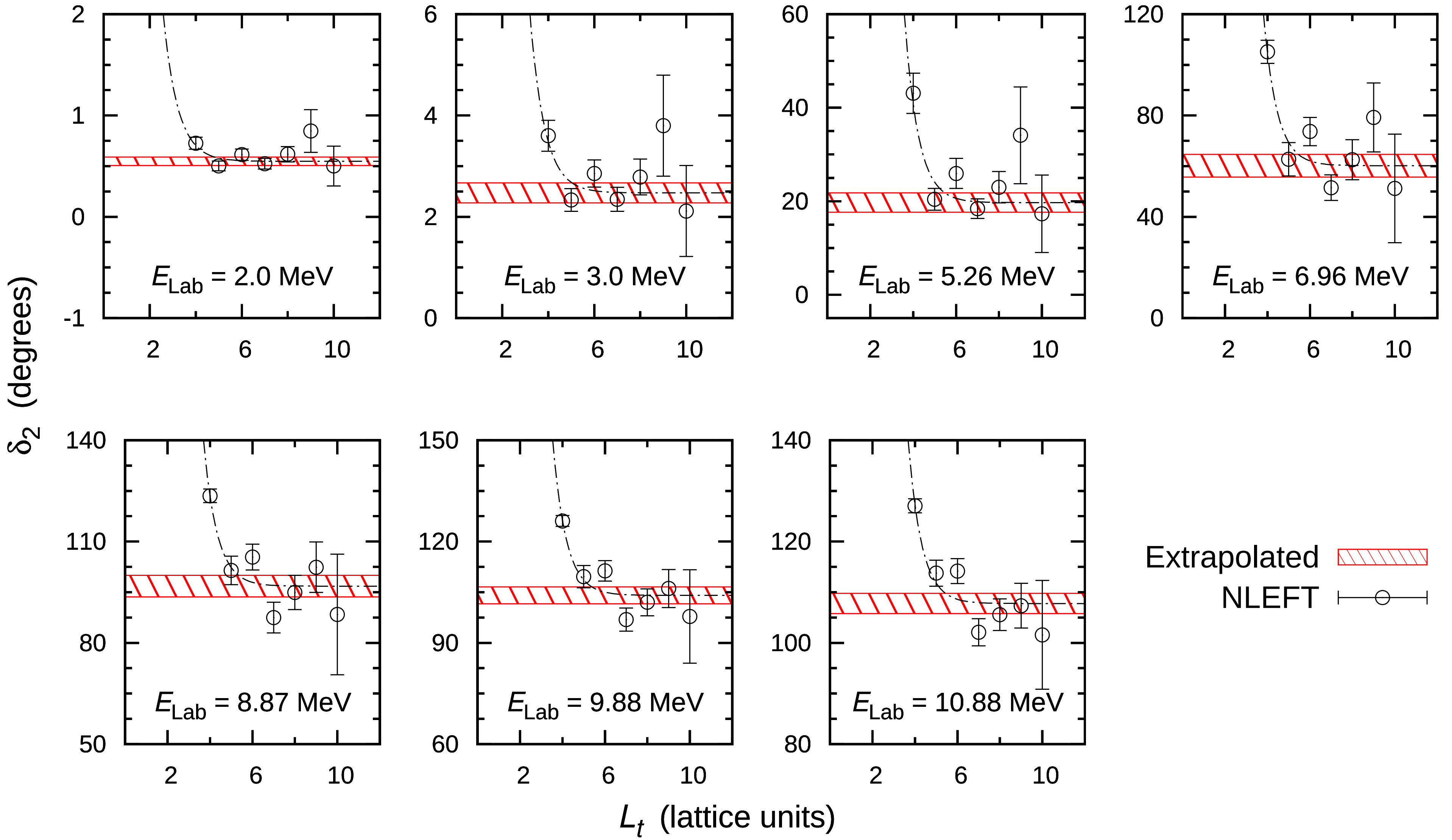
LATTICE DATA I

- Show data for the S-wave:



LATTICE DATA II

- Show data for the D-wave:



LOCAL/NON–LOCAL INTERACTIONS on the LATTICE

- Local operators/densities:

  $a(\mathbf{n}), a^\dagger(\mathbf{n})$ [$\mathbf{n}$ denotes a lattice point]

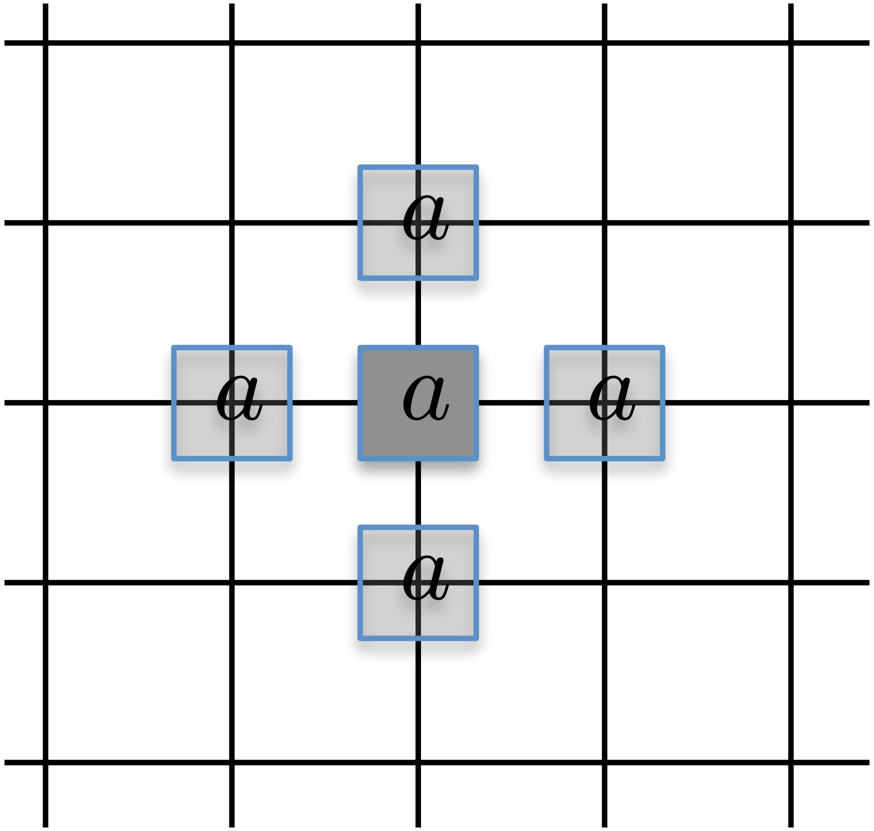
  $$\rho_\mathrm{L}(\mathbf{n}) = a^\dagger(\mathbf{n}) a(\mathbf{n})$$

- Non-local operators/densities:

  $$a_\mathrm{NL}(\mathbf{n}) = a(\mathbf{n}) + s_\mathrm{NL} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a(\mathbf{n}')$$

  $$a^\dagger_\mathrm{NL}(\mathbf{n}) = a^\dagger(\mathbf{n}) + s_\mathrm{NL} \sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')$$

  $$\rho_\mathrm{NL}(\mathbf{n}) = a^\dagger_\mathrm{NL}(\mathbf{n}) a_\mathrm{NL}(\mathbf{n})$$

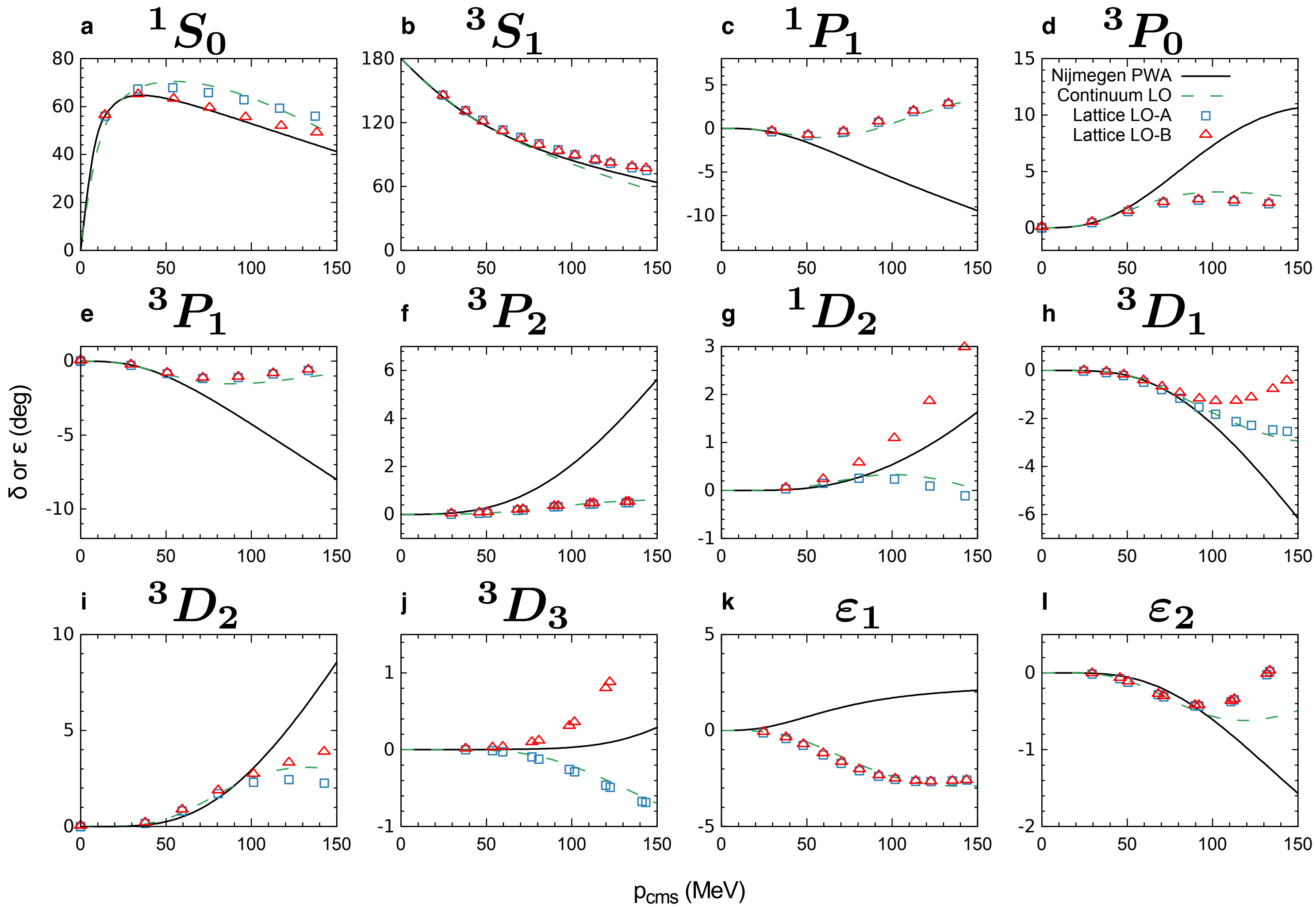


$\rightarrow$ where $\sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle}$ denotes the sum over nearest-neighbor lattice sites of $\mathbf{n}$

$\rightarrow$ the smearing parameter s_NL is determined when fitting to the phase shifts

NUCLEON–NUCLEON PHASE SHIFTS

- Show results for NN [and α-α] phase shifts for both interactions:



→ both interactions very similar

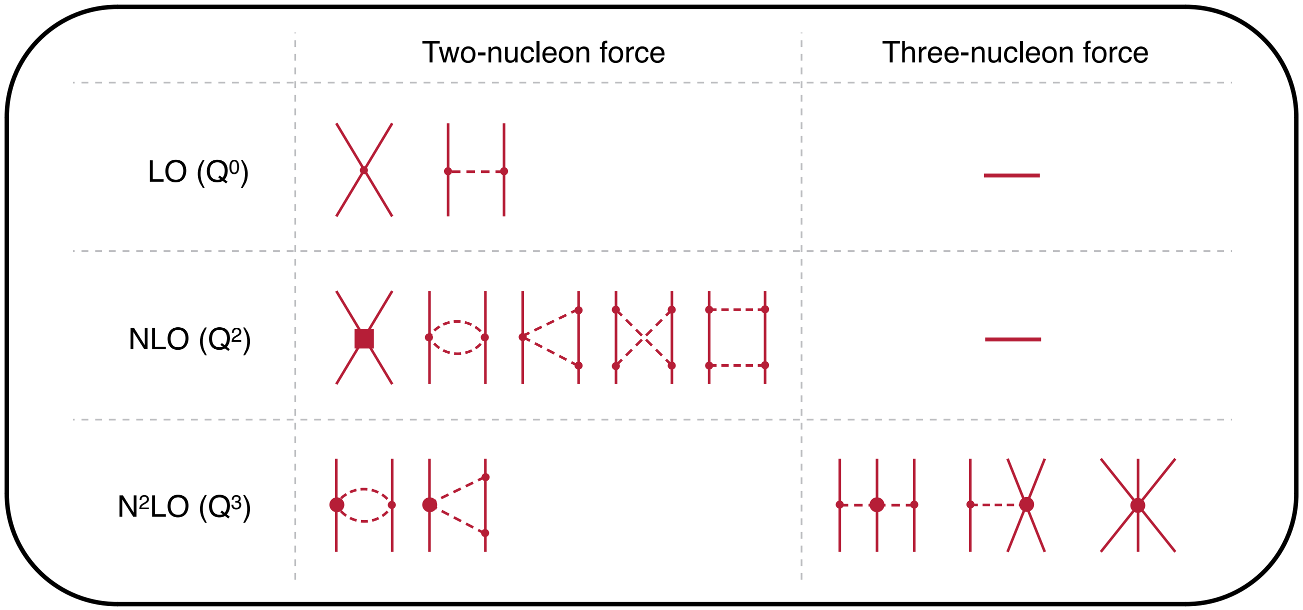
Neutron-proton scattering at NNLO for varying lattice spacings

Alarcón, Du, Klein, Lähde, Lee, Li, Luu, UGM
Eur. Phys. J. **A** (2017) in print [arXiv:1702.05319]

NUCLEAR FORCES at NNLO

for details, see: Epelbaum, Hammer, UGM, Rev. Mod. Phys. **81** (2009) 1773

- Potential at next-to-next-to-leading order $[Q = \{p/\Lambda, M_\pi/\Lambda\}]$:



- NN potential to NNLO [all πN and $\pi\pi N$ LECs fixed from πN scattering]:

$$V_{\rm NN} = V_{\rm LO}^{(0)} + V_{\rm NLO}^{(2)} + V_{\rm NNLO}^{(3)}$$

$$= V_{\rm LO}^{\rm cont} + V_{\rm LO}^{\rm OPE} + V_{\rm NLO}^{\rm cont} + V_{\rm NLO}^{\rm TPE} + V_{\rm NNLO}^{\rm TPE}$$

NUCLEAR FORCES at NNLO continued

- Analytic expressions [2+7 LECs]:

$$V_{\rm LO}^{\rm cont} = C_S + C_T\,(\vec\sigma_1\cdot\vec\sigma_2)$$

$$V_{\rm LO}^{\rm OPE} = -\frac{g_A^2}{4F_\pi^2}\,\tau_1\cdot\tau_2\,\frac{(\vec\sigma_1\cdot\vec q\,)\,(\vec\sigma_2\cdot\vec q\,)}{q^{\,2}+M_\pi^2}$$

$\vec q$ = t-channel mom. transfer

$$V_{\rm NLO}^{\rm cont} = C_1 q^2 + C_2 k^2 + \left(C_3 q^2 + C_4 k^2\right)(\vec\sigma_1\cdot\vec\sigma_2) + iC_5\tfrac{1}{2}\,(\vec\sigma_1+\vec\sigma_2)\cdot(\vec q\times\vec k\,)$$
$$+\, C_6\,(\vec\sigma_1\cdot\vec q\,)\,(\vec\sigma_2\cdot\vec q\,) + C_7(\vec\sigma_1\cdot\vec k\,)(\vec\sigma_2\cdot\vec k\,)$$

$\vec k$ = u-channel mom. transfer

$$V_{\rm NLO}^{\rm TPE} = -\tfrac{\tau_1\cdot\tau_2}{384\pi^2F_\pi^4}L(q)\Big[4M_\pi^2\left(5g_A^4-4g_A^2-1\right)+q^2\left(23g_A^4-10g_A^2-1\right)$$
$$+\,\tfrac{48g_A^4M_\pi^4}{4M_\pi^2+q^2}\Big] - \tfrac{3g_A^4}{64\pi^2F_\pi^4}L(q)\left[(\vec q\cdot\vec\sigma_1)\,(\vec q\cdot\vec\sigma_2) - q^2\,(\vec\sigma_1\cdot\vec\sigma_2)\right]$$

- Loop function: $$L(q) = \tfrac{1}{2q}\sqrt{4M_\pi^2+q^2}\,\ln\frac{\sqrt{4M_\pi^2+q^2}+q}{\sqrt{4M_\pi^2+q^2}-q}$$
$$\to 1 + \tfrac{1}{3}\,\tfrac{q^2}{4M_\pi^2} + \cdots \text{ for } q \ll \Lambda$$

$\to$ for coarse lattices $a \simeq 2$ fm, the TPE at N(N)LO can be absorbed in the LECs C_i

$\to$ no longer true as a decreases, need to account for the TPE explicitely

A FEW DETAILS ON THE FITS

- Fits in large & fixed volumes, vary a from 1 to 2 fm:

a^{-1} [MeV]	a [fm]	L	La [fm]
100	1.97	32	63.14
120	1.64	38	62.48
150	1.32	48	63.14
200	0.98	64	63.14

- OPE and TPE LECs completely fixed ($g_A \sim g_{\pi NN}$ and $c_{1,2,3,4}$ from RS analysis)

Hoferichter, Ruiz de Elvira, Kubis, UGM, Phys. Rev. Lett. **115** (2015) 092301

- Smeared LO S-wave contact interactions: $f(\vec{q}) \equiv f_0^{-1} \exp\left(-b_s \frac{\vec{q}^{\,4}}{4}\right)$

- Partial-wave projection of the contact interactions

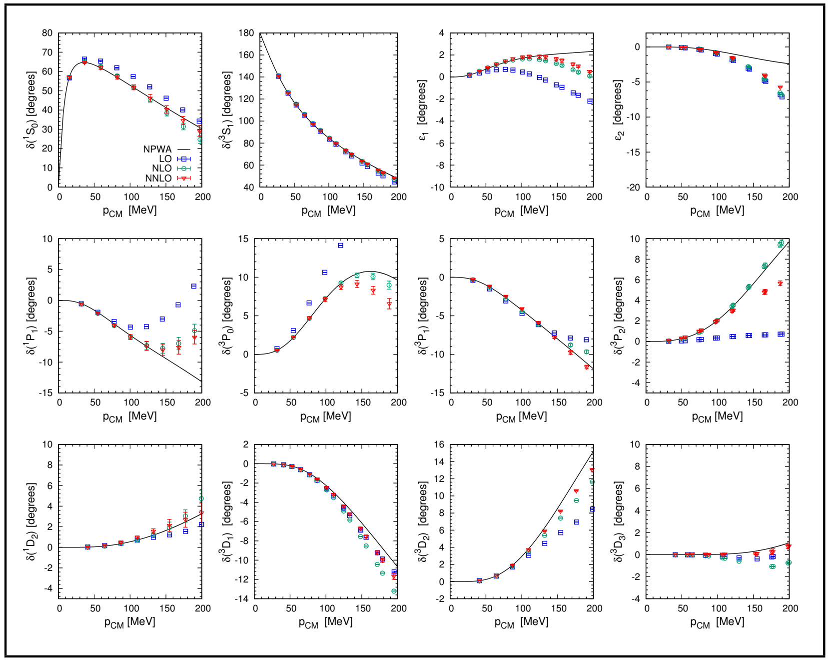
→ fit b_s and two S-wave LECs C_i at LO up to $p_{\rm cm} = 100$ MeV

→ w/ b_s fixed, fit two/seven S/P-wave LECs C_i at NLO/NNLO up to $p_{\rm cm} = 150$ MeV

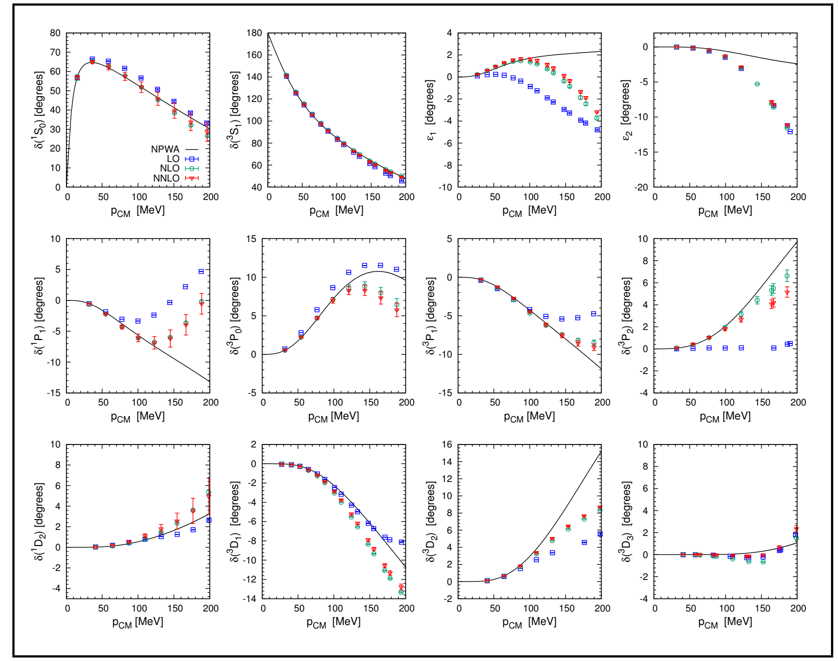
→ treat NLO and NNLO corrections perturbatively and non-perturbatively

RESULTS for VARIOUS LATTICE SPACINGS - nonpert.

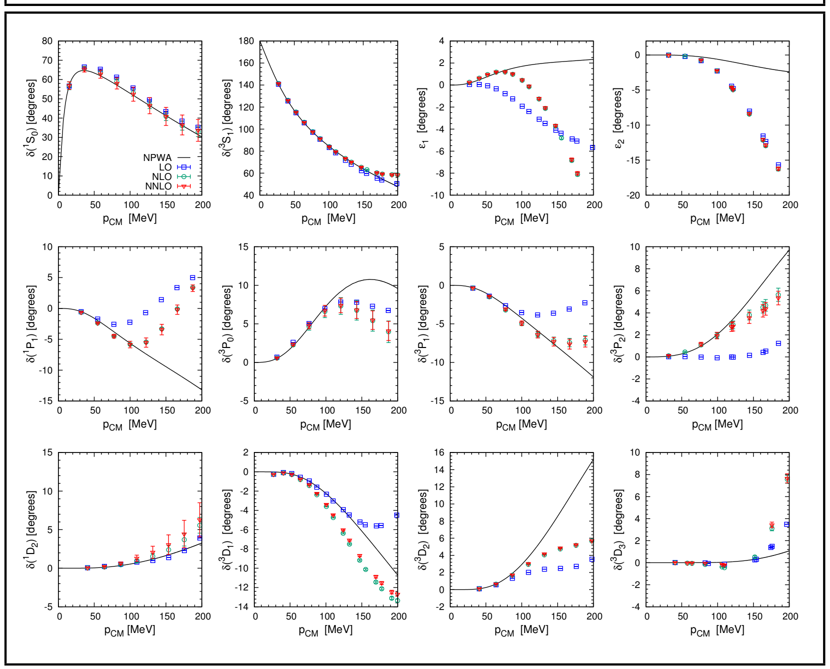
$a = 0.98$ fm



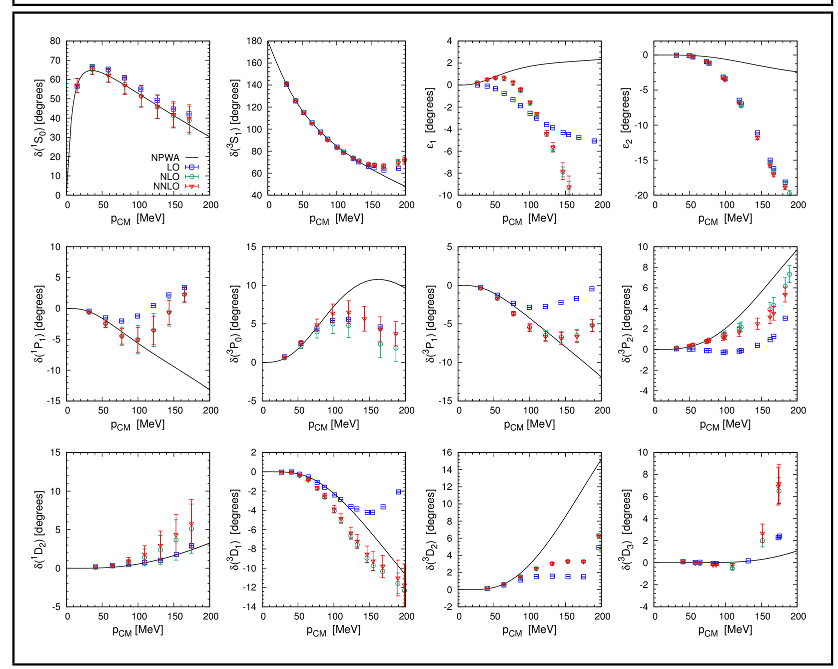
$a = 1.32$ fm



$a = 1.64$ fm



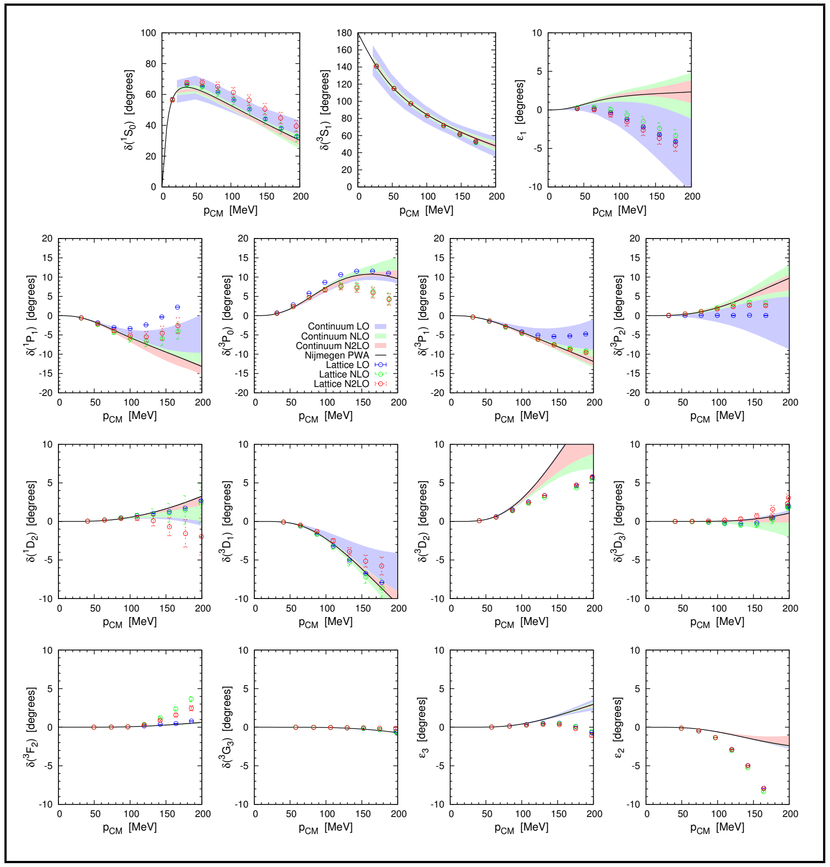
$a = 1.97$ fm



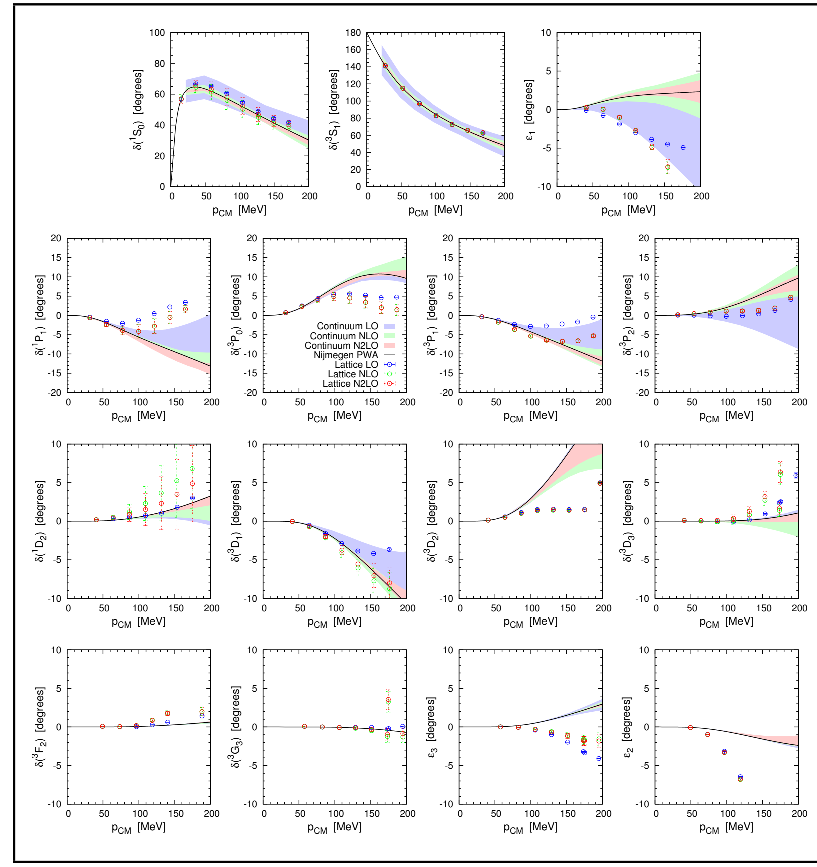
RESULTS for VARIOUS LATTICE SPACINGS - pert.

- perturbative treatment of NLO and NNLO corrections

$a = 1.32$ fm



$a = 1.97$ fm



$\rightarrow$ up to $p_{\rm cm} \simeq 150$ MeV, physics is indendependent of a $\surd$

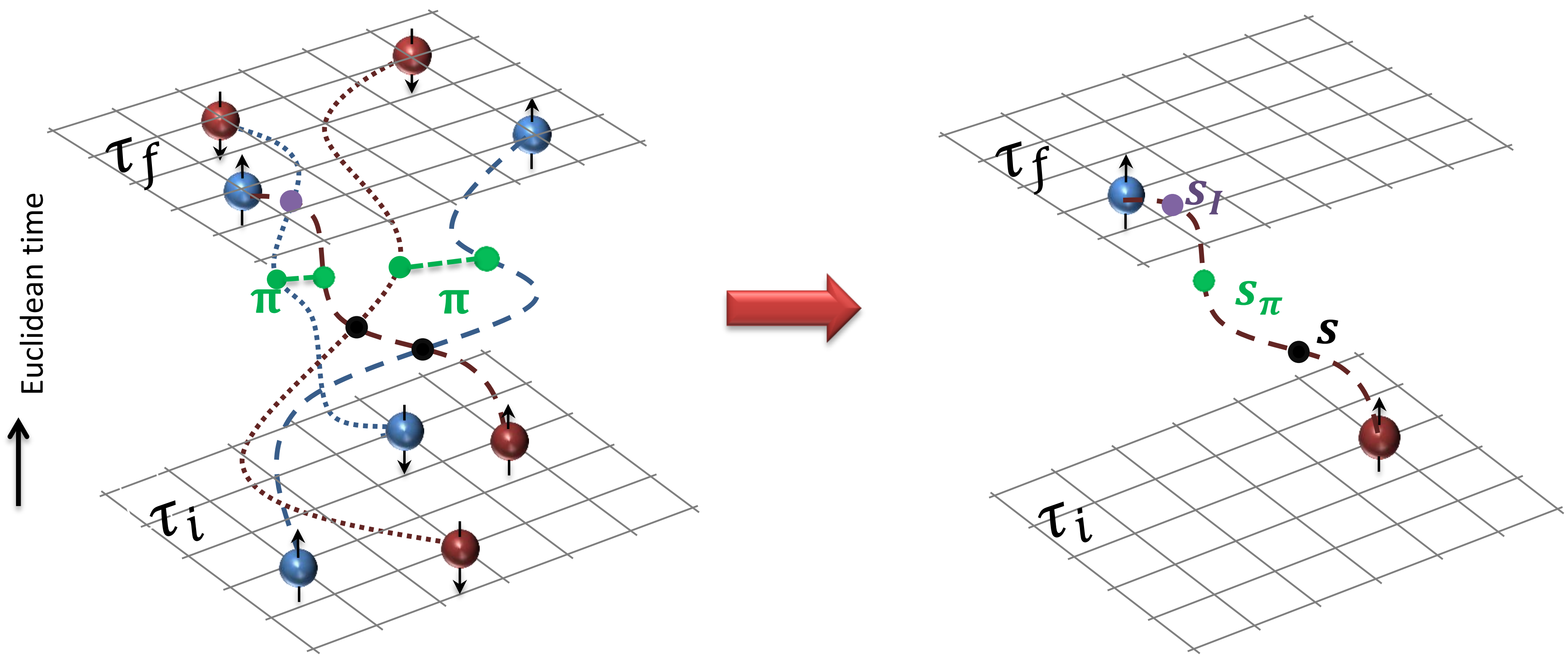
$\rightarrow$ description consistent with the continuum within error bands $\surd$

$\rightarrow$ explore this for nuclei —- work in progress / stay tuned

AUXILIARY FIELD METHOD

- Represent interactions by auxiliary fields:

$$\exp\left[-\frac{C}{2}\left(N^\dagger N\right)^2\right] = \sqrt{\frac{1}{2\pi}}\int ds \exp\left[-\frac{s^2}{2} + \sqrt{C}\, s\left(N^\dagger N\right)\right]$$



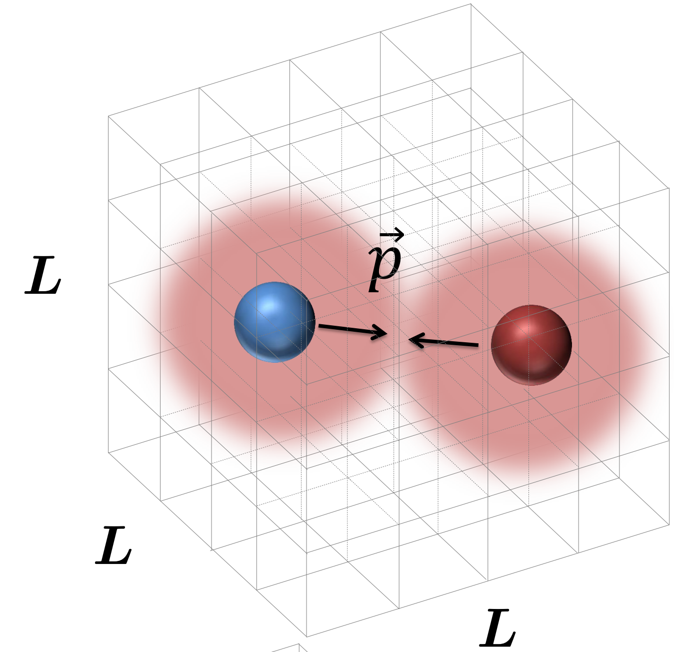
EXTRACTING PHASE SHIFTS on the LATTICE

- Lüscher's method:

  Two-body energy levels below the inelastic threshold on a periodic lattice are related to the phase shifts in the continuum

  Lüscher, Comm. Math. Phys. **105** (1986) 153

  Lüscher, Nucl. Phys. B **354** (1991) 531



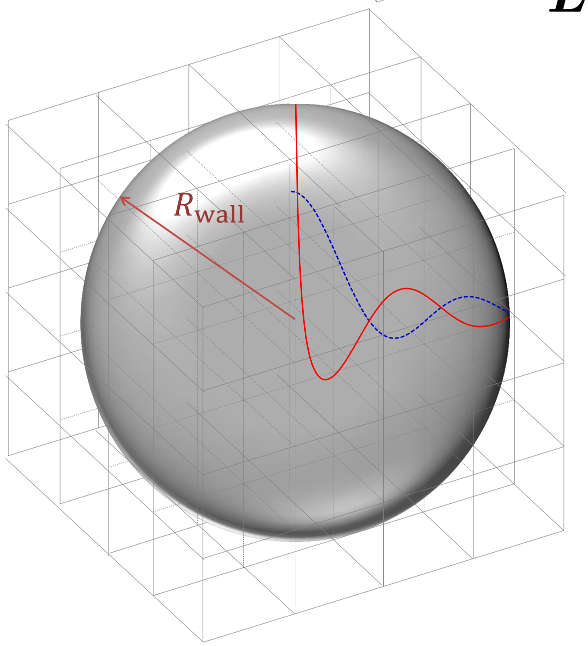
- Spherical wall method:

  Impose a hard wall on the lattice and use the fact that the wave function vanishes for $r = R_{\rm wall}$:

$$\psi_\ell(r) \sim [\cos\delta_\ell(p) F_\ell(pr) + \sin\delta_\ell(p) G_\ell(pr)]$$

  Borasoy, Epelbaum, Krebs, Lee, UGM, EPJA **34** (2007) 185

  Carlson, Pandharipande, Wiringa, NPA **424** (1984) 47



ADIABATIC HAMILTONIAN

- Construct the adiabatic Hamiltonian from the dressed cluster states:

$$[H_\tau]_{\vec{R}\vec{R}'} = {}_\tau\langle\vec{R}|H|\vec{R}'\rangle_\tau$$

- States are i.g. not normalized, require *norm matrix*:

$$[N_\tau]_{\vec{R}\vec{R}'} = {}_\tau\langle\vec{R}|\vec{R}'\rangle_\tau$$

- construct the full adiabatic Hamiltonian:

$$\left[H^a_\tau\right]_{\vec{R}\vec{R}'} = \sum_{\vec{R}_n\vec{R}_m} \left[N_\tau^{-1/2}\right]_{\vec{R}\vec{R}_n} \left[H_\tau\right]_{\vec{R}_n\vec{R}_m} \left[N_\tau^{-1/2}\right]_{\vec{R}_m\vec{R}'}$$

$\rightarrow$ The structure of the adiabatic Hamiltonian is similar to the Hamiltonian matrix used in recent ab initio NCSM/RGM calculations

Navratil, Quaglioni, Phys. Rev. C **83** (2011) 044609
Navratil, Roth, Quaglioni, Phys. Lett. B **704** (2011) 379
Navratil, Quaglioni, Phys. Rev. Lett. **108** (2012) 042503

TESTING the ADIABATIC HAMILTONIAN

- Consider fermion-dimer scattering:

Microscopic Hamiltonian $\Rightarrow$ Two-cluster adiabatic Hamitltonian

$$L^{3(A-1)} \times L^{3(A-1)} \qquad \qquad L^3 \times L^3$$

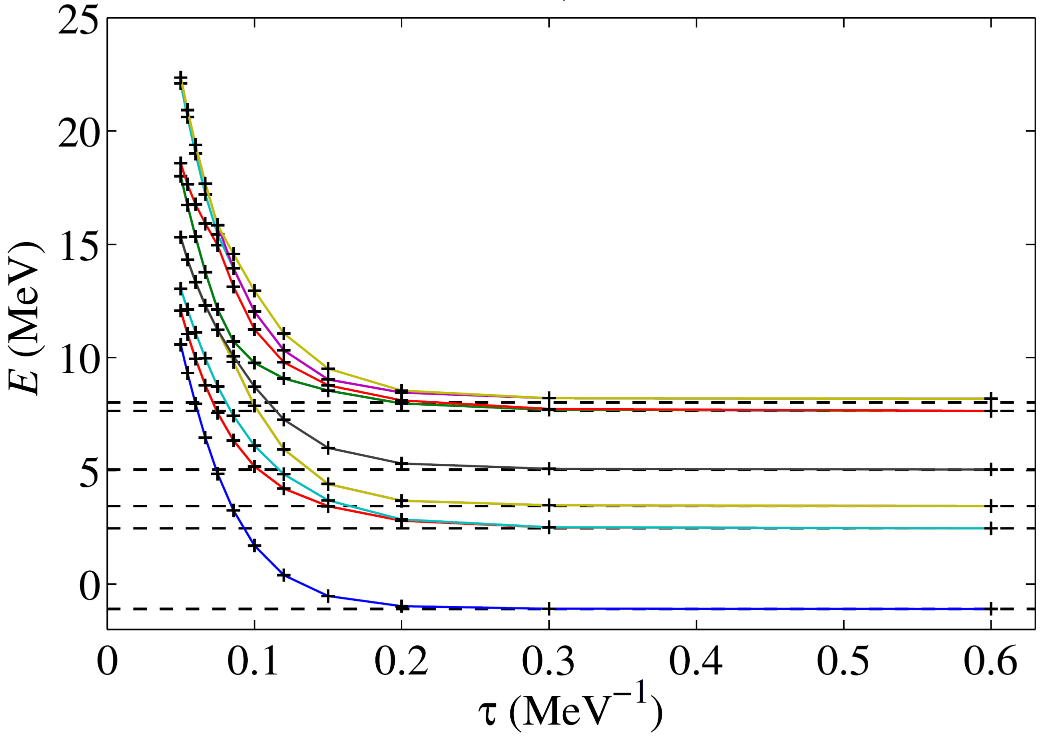
- calculation of a 7^3 lattice, lattice spacing $a = 1.97$ fm

Pine, Lee, Rupak, EPJA **49** (2013) 151

exact Lanzcos: black dashed lines

adiab. Ham.: solid colored lines

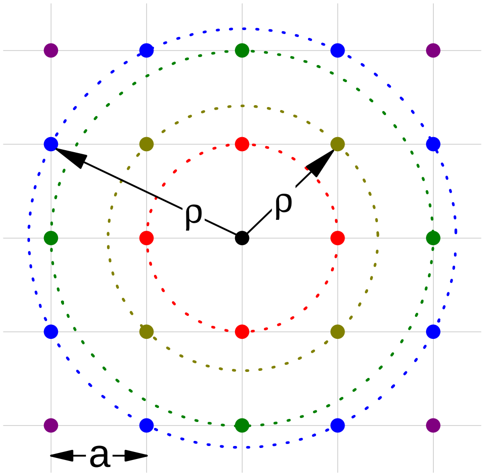
lattice size is 7^3 , $a = 1.93$ fm



ALPHA–ALPHA SCATTERING

- same lattice action as for the Hoyle state in ^{12}C and the structure of ^{16}O
- (9+2) NN + 2 3N LECs, coarse lattice $a = 1.97$ fm, $N = 8$
- new algorithm for Monte Carlo updates and alpha clusters
- adiabatic projection method to construct a two-alpha Hamiltonian
- spherical wall method to extract the phase shifts using radial Hamiltonian

$$|R\rangle^{\ell,\ell_z} = \sum_{\vec{R}'} Y_{\ell,\ell_z}(\vec{R}')\,\delta_{R,|\vec{R}'|}\,|\vec{R}'\rangle$$



$\rightarrow$ precise extraction of phase shifts & mixing angles

Lu, Lähde, Lee, UGM, Phys. Lett. B **760** (2016) 309

Moinard et al., work in progress

Elhatisari, Lee, UGM, Rupak, Eur. Phys. J. A **52** (2016) 174